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Marius I. Stoka and Giuseppe Caristi

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On a problem of Geometric probabilities in the Euclidean Space

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Abstract

In this note we compute the probability $p_{\S,\mathcal{R}}$ that a random segment S whose length l is a bounded random variable Δ, and known moments $E(\Delta)$, $E(\Delta^2)$, $E(\Delta^3)$, intersects a lattice $\mathcal{R}$ obtained by partitioning $\mathbf{E}_3$ whit a family of elementary tiles consists of a right prism, of height c, whose base is a right angled triangle of catheti a and $b = a \tan\alpha$ $(0 < \alpha < \pi/2)$.

AMS Subject Classification: 60D05, 52A22
Key Words: Integral geometry. Lie groups. Invariant varieties.

1 Introduction

In 1733, at a meeting of the Académie des Sciences de Paris, Buffon posed a problem that later on should become known as the famous "*Buffon needle problem*": in a room, the floor of which is merely divided by parallel lines, at a distance a apart, a needle of length $l < a$ is allowed to fall at random: which is the probability that the needle intersects one of the lines?

The solution, determined by Buffon by means of empirical methods, was $p = 2l/\pi a$. The problem and its solution were published in 1777, in the "*Comptes rendus de l'Académie des sciences de Paris*".

In 1812, Laplace extended the problem by considering a room paved with equal tiles, shaped as rectangles of sides a and b, with $l < min(a, b)$. The solution was $p = \frac{2l(a+b)-l^2}{\pi ab}$, and it is obvious that the probability of Buffon can be obtained from that of Laplace by letting $b \to +\infty$.

We restate now these problems in a slightly different form, which will be useful for several different extensions. In the classical problem of Buffon type, we have always referred to a test body $\mathbf{T}$, uniformly distributed in the Euclidean space. Obviously all the results are essentially tied to our particular definition of random positioning of a body in the space and to the choice of the elementary Kinematic measure which provides invariance of probability statements under a special group of transformations. Other definitions are possible and these lead to different results. For example we might need to anisotropically distributed test bodies (see e.g. [1], [3]), or we can find that invariance under rotation and translation is too demanding, and we can relax this condition (see [2]). In this order of idea in the paper in this note we compute the probability $p_{\S,\mathcal{R}}$ that a random segment S whose length l is a bounded random variable Δ, and known moments $E(\Delta)$, $E(\Delta^2)$, $E(\Delta^3)$, intersects a lattice $\mathcal{R}$ obtained by partitioning $\mathbf{E}_3$ whit a family of elementary tiles consists of a right prism, of height c, whose base is a right angled triangle of catheti a and $b = a\tan\alpha$ $(0 < \alpha < \pi/2)$.

2 The Lattice $\mathcal{R}$

Let us denote by $\mathbf{E}_3$ be the Euclidean plane. By a lattice $\mathcal{R}$ in $\mathbf{E}_3$ we understand a sequence of closed and connected sets $\{\mathcal{C}_n\}_{n\in N}$ such that

1. $\bigcup_{n\in N}\mathcal{C}_n = \mathbf{E}_3$,
2. $Int(\mathcal{C}_i)\cap Int(\mathcal{C}_j) = \emptyset$, $\forall i, j \in N$ and $i \neq j$,
3. $\mathcal{C}_n = \gamma_n(\mathcal{C}_0)$, $\forall n \in N$, where γ_n are the elements of a discrete subgroup of the group of motions in $\mathbf{E}_3$ that leaves invariant the lattice.

The domain $\mathcal{C}_0$ is usually called the *fundamental tile* (or cell) of $\mathcal{R}$. Let us denote by $\mathbf{K}$ be a convex body (which means here a compact convex set) which we shall call *test body*. A general problem of Buffon type can be stated as follows:

"*Which is the probability $p_{\mathbf{K},\mathcal{R}}$ that the random convex body $\mathbf{K}$, or more precisely, the random congruent copy of $\mathbf{K}$, meets some of the boundary points of at least one of the domains $\mathcal{C}_n$?* "

In the following part of the paper we shall deal with a lattice $\mathcal{R}$ obtaining $\mathbf{E}_3$ where each elementary tile consists of a right prism, of height c, whose base is a right angled triangle of catheti a and $b = a\tan\alpha$ $(0 < \alpha < \pi/2)$. Let

$$\mathcal{C}_0(\mathcal{R}) := \left\{(x, y, z) \in \mathbf{E}_3 : 0 \le x \le a, 0 \le y \le (a - x)\tan\alpha, 0 \le z \le c\right\},$$

Then $\mathcal{R}$ is obtained as

$$\mathcal{R} := \mathcal{R}^1 \cup \mathcal{R}^2 \cup \mathcal{R}^3 \cup \mathcal{R}^4,$$

where the elementary lattices $\mathcal{R}^i$ are defined as follows

$$\mathcal{R}^1 := \Big\{x = na, n \in \mathbf{Z}\Big\},$$

$$\mathcal{R}^2 := \Big\{y = nb, n \in \mathbf{Z}\Big\},$$

$$\mathcal{R}^3 := \Big\{x - z\cot\xi - na\sin\alpha = 0, n \in \mathbf{Z}\Big\},$$

$$\mathcal{R}^4 := \Big\{y = (a - x)n\tan\alpha, n \in \mathbf{Z}\Big\},$$

3 Main Result

With the notations of before we obtain the following result

Theorem 1. *The probability $p_{S,\mathcal{R}}$ that a random segment S whose length l is a bounded random variable Δ, with upper boundary $l < \min(b\cos\alpha, c)$, and known moments $E(\Delta)$, $E(\Delta^2)$, $E(\Delta^3)$, intersects the lattice $\mathcal{R}$ is the following:*

$$p_{S,\mathcal{R}} = \frac{ab + (1+\cos\alpha)ac + (1+\sin\alpha)bc}{2abc}E(\Delta) - \frac{1}{3\pi abc}\Lambda_1 E(\Delta^2) + \Lambda_2 E(\Delta^3),$$

where we put

$$\Lambda_1 := 2(1+\cos\alpha)a + 2(1+\sin\alpha)b + \Big[3 + (\pi - \alpha)\cot\alpha + \Big(\frac{\pi}{2} + \alpha\Big)\tan\alpha\Big]c,$$

and

$$\Lambda_2 := \frac{3 + (\pi - \alpha)\cot\alpha + (\pi/2 + \alpha)}{8\pi abc}.$$

Proof. The upper boundary of Δ assures that S is always small when compared to the elementary tile of $\mathcal{R}$. Let $f(\delta)$ be the density of Δ and $p(\mathcal{R}|\delta)$ the probability that S intersects a cell of $\mathcal{R}$ given the condition $\Delta = \delta$. Then the probability $p_{S,\mathcal{R}}$ that S intersects at least one of the planes of the lattice can be computed as

$$p_{S,\mathcal{R}} = \int_0^D p(\mathcal{R}|\delta)f(\delta)d\delta.$$

Hence, a long computation ensures that

$$p_{S,\mathcal{R}} = \frac{ab + (1+\cos\alpha)ac + (1+\sin\alpha)bc}{2abc} \int_0^D \delta f(\delta)d\delta - \frac{1}{3\pi abc}\Lambda_1 \int_0^D \delta^2 f(\delta)d\delta +$$

$$+\Lambda_2 \int_0^D \delta^3 f(\delta)d\delta,$$

which is the stated formula.

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[1] A. Duma, A. Rizzo, The Buffon needle problem in a magnetic field, *Suppl. Rend. Circ. Mat. Palermo,* serie II, 35, 1994, pp. 127-142.

[2] H. Solomon, Geometric Probability, *Soc. for Ind. and Appl. Math.,* 1978.

[3] M. Stoka, Il problema dell'ago di Buffon nella teoria delle probabilità geometriche, *Conf. Semin. Mat. Univ. Bari,* 190, 1983.

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On a Buffon's Problem

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Abstract

In this note we compute the probability $p_{\Sigma,\mathcal{R}}$ that a random sphere Σ whose diameter D is a bounded random variable Δ, and known moments $E(\Delta)$, $E(\Delta^2)$, $E(\Delta^3)$, intersects a lattice $\mathcal{R}$ of parallelepipeds of dimensions (a, b, α), (b, c, β), (a, c, γ).

AMS Subject Classification: 60D05, 52A22
Key Words: Integral geometry. Lie groups. Invariant varieties.

1 Introduction

In the classical problem of Buffon type, we have always referred to a test body **T**, uniformly distributed in the Euclidean space. Obviously all the results are essentially tied to our particular definition of random positioning of a body in the space and to the choice of the elementary Kinematic measure which provides invariance of probability statements under a special group of transformations. Other definitions are possible and these lead to different results. For example we might need to anisotropically distributed test bodies (see e.g. [1], [3]), or we can find that invariance under rotation and translation is too demanding, and we can relax this condition (see [2]). In this order of idea in the paper we compute the probability $p_{S,\mathcal{R}}$ that a random sphere Σ whose diameter D is a bounded random variable Δ, and known moments $E(\Delta)$, $E(\Delta^2)$, $E(\Delta^3)$, intersects a lattice $\mathcal{R}$ of parallelepipeds of dimensions (a, b, α), (b, c, β), (a, c, γ).

2 The Lattice $\mathcal{R}$

Let us denote by $\mathbf{E}_3$ be the Euclidean plane. By a lattice $\mathcal{R}$ in $\mathbf{E}_3$ we understand a sequence of closed and connected sets $\{\mathcal{C}_n\}_{n\in N}$ such that

1. $\bigcup_{n\in N}\mathcal{C}_n = \mathbf{E}_3$,
2. $Int(\mathcal{C}_i)\cap Int(\mathcal{C}_j)=\emptyset$, $\forall i,j\in N$ and $i\neq j$,
3. $\mathcal{C}_n=\gamma_n(\mathcal{C}_0)$, $\forall n\in N$, where γ_n are the elements of a discrete subgroup of the group of motions in $\mathbf{E}_3$ that leaves invariant the lattice.

The domain $\mathcal{C}_0$ is usually called the *fundamental tile* (or cell) of $\mathcal{R}$. Let us denote by $\mathbf{K}$ be a convex body (which means here a compact convex set) which we shall call *test body.* A general problem of Buffon type can be stated as follows:

"*Which is the probability $p_{\mathbf{K},\mathcal{R}}$ that the random convex body $\mathbf{K}$, or more precisely, the random congruent copy of $\mathbf{K}$, meets some of the boundary points of at least one of the domains $\mathcal{C}_n$?* "

In the following part of the paper we shall deal with a lattice $\mathcal{R}$ obtaining $\mathbf{E}_3$ with a family of oblique parallelepipeds congruent to the elementary tile:

$$\mathcal{R}:=\Bigg\{(x,y,z)\in\mathbf{E}_3: 0\le x\le \cot\xi z-a\sin\alpha,$$

$$x\cot\alpha+\frac{\cot\xi}{\sin\alpha}z\le y\le b+x\cot\alpha+\frac{\cot\xi}{\sin\alpha}z, 0\le z\le c\sin\gamma\sin\xi\Bigg\},$$

where $\xi,\gamma,\alpha\in(0,\pi/2)$ and $a,b,c\in\mathbf{R}_+^*$. (i.e. a lattice of parallelepipeds of dimensions (a,b,α), (b,c,β), (a,c,γ))

Once again we have the following decomposition

$$\mathcal{R}:=\mathcal{R}^1\cup\mathcal{R}^2\cup\mathcal{R}^3,$$

where the elementary lattices $\mathcal{R}^i$ are defined as follows

$$\mathcal{R}^1:=\Big\{x-z\cot\xi-na\sin\alpha=0, n\in\mathbf{Z}\Big\},$$

$$\mathcal{R}^2:=\Big\{x\cot\alpha-y+z\frac{\cot\xi}{\sin\alpha}+nb=0, n\in\mathbf{Z}\Big\},$$

$$\mathcal{R}^3:=\Big\{z=nc\sin\gamma\sin\xi, n\in\mathbf{Z}\Big\},$$

Easy geometrical considerations prove that the acute angles α, β and γ are determined by the conditions

$$\cos\beta = \cos\alpha = \cos\gamma = \frac{\cos\xi + \cos^2\xi}{\sin^2\xi}.$$

We put

$$d_1 := a\sin\alpha\sin\xi,$$

$$d_2 := b\sin\beta\sin\xi,$$

$$d_3 := c\sin\gamma\sin\xi.$$

3 Main Result

With the notations of before we obtain the following result

Theorem 1. *The probability $p_{S,\mathcal{R}}$ that a random segment S whose length l is a bounded random variable Δ, with upper boundary $l < \min(d_1, d_2, d_3)$, and known moments $E(\Delta)$, $E(\Delta^2)$, $E(\Delta^3)$, intersects the lattice $\mathcal{R}$ is the following:*

$$p_{S,\mathcal{R}} = \frac{d_1d_2 + d_1d_3 + d_2d_3}{d_1d_2d_3}E(\Delta) - \Psi_1(d_1, d_2, d_3)E(\Delta^2) + \\ + \Psi_2(d_1, d_2, d_3)E(\Delta^3),$$

where we put

$$\Psi_1(\xi, d_1, d_2, d_3) := \frac{(d_1 + d_2 + d_3)}{d_1d_2d_3},$$

and

$$\Psi_2(\xi, d_1, d_2, d_3) := \frac{1}{d_1d_2d_3}.$$

Proof. The upper boundary of Δ assures that S is always small when compared to the elementary tile of $\mathcal{R}$. Let $f(\delta)$ be the density of Δ and $p(\mathcal{R}|\delta)$ the probability that S intersects a cell of $\mathcal{R}$ given the condition $\Delta = \delta$. Then the probability $p_{S,\mathcal{R}}$ that S intersects at least one of the planes of the lattice can be computed as

$$p_{S,\mathcal{R}} = \int_0^D p(\mathcal{R}|\delta)f(\delta)d\delta.$$

Hence, a long computation ensures that

$$p_{\Sigma,\mathcal{R}} = \frac{d_1d_2 + d_1d_3 + d_2d_3}{d_1d_2d_3} \int_0^D \delta f(\delta)d\delta - \Psi_1(\xi, d_1, d_2, d_3) \int_0^D \delta^2 f(\delta)d\delta +$$

$$+\Psi_2(\xi, d_1, d_2, d_3) \int_0^D \delta^3 f(\delta)d\delta,$$

which is the stated formula.

Remark. If $\alpha = \beta = \gamma = \pi/2$ we obtain a generalization of a classical result of Stoka (see [4]).

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[1] A. Duma, A. Rizzo, The Buffon needle problem in a magnetic field, *Suppl. Rend. Circ. Mat. Palermo,* serie II, 35, 1994, pp. 127-142.

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Estimating the support of a probability distribution

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Abstract

This paper deals with estimation of the support S of an unknown probability distribution from observations. First, we briefly recall some known results. Then, we study rate of convergence of the Geffroy estimator of the edge function g when S is not compact. We also construct and study a smooth estimator of g based on a nonparametric density estimator.

Key words: Estimation, Support of a probability distribution, Estimation of edge function, density estimator.

1 Introduction

Estimation of the support of an unknown probability distribution based on a sample from this distribution is useful in various applications : image reconstruction, detection of oilfield, efficiency of enterprises etc ...

In this paper we give a brief survey of that topic and derive some new asymptotical results. We also discuss criteria of efficiency for support estimators.

The precise statement of our problem is as follows : let $Z_1, \dots, Z_n$ be independent equidistributed observed random variables with values in a separable metric space E equipped with its Borel σ-algebra $\mathcal{B}_E$.

Let μ be the distribution of Z_i, its support S is defined as

$$S = \Big\{ z \in E,\ \forall \varepsilon > 0,\ \mu\Big(B(z, \varepsilon)\Big) > 0 \Big\},$$

Email address: bosq@ccr.jussieu.fr (Denis Bosq).

where $B(z, \varepsilon)$ is the open ball with center z and radius ε. S is also the smallest closed set with probability 1.

The problem is to construct a suitable estimator $S_n = S_n(Z_1, \ldots, Z_n)$ of S.

Some important criteria of quality for S_n are the following :

- Similarity of the *shapes* of S_n and S.
- Optimal *rate* of convergence with respect to various topologies.
- *Easiness* of computation.
- *Absence of systematic bias.*

A suitable estimator should be a compromise between these qualities. Comments concerning suitability of some support estimators will appear below.

The paper is organized as follows : Section 2 recalls some results concerning estimation of S in the general and convex cases. Estimation of the edge function of S is considered in Section 3. In particular we derive rate of convergence if S is not bounded. Section 4 deals with support estimators based on density estimators. We present various methods and construct a smooth estimator of the edge function. Some proofs appear in Section 5.

Finally note that limit in distribution is not considered in this paper. For this topic the reader is referred to Geffroy (1964); Geffroy et al. (2006) and Girard and Jacob (2007) who propose an original estimation method.

2 The general and convex cases

In this section we suppose that $(E, \mathcal{B}_E) = (\mathbb{R}^d, \mathcal{B}_{\mathbb{R}^d})$ $(d \geq 1)$ with its euclidian norm $\|\cdot\|$. Then a natural estimator of S is given by

$$S_n = \bigcup_{i=1}^{n} \left\{ z \ : \ \|Z_i - z\| \leq h_n \right\}$$

where $\lim_{n \to \infty} h_n = 0\ (+)$.

This estimator was proposed by Chevalier (1976). If S is compact with a boundary δS piecewise Lipschitz then, under a "core condition", Korostolev and Tsybakov (KT) (1993) have shown that the choice $h_n = c\left(\frac{\ln n}{n}\right)^{1/d}$, where c is a suitable constant, leads to

$$\text{(1)} \qquad \mathrm{E}\left(\ell^d(S \Delta S_n)\right) = \mathcal{O}\left(\left(\frac{\ln n}{n}\right)^{1/d}\right)$$

where ℓ is Lebesgue measure and Δ is symmetric difference. This rate is optimal in a minimax sense. Note that S_n is easy to construct but its shape is not satisfactory if some information concerning the shape of S is available.

In particular, if S is a compact convex set one may improve the rate in (1) by considering the convex hull $\widehat{S}_n$ of $\{Z_1, \dots, Z_n\}$. If μ is the uniform distribution over S, $\widehat{S}_n$ is nothing but the maximum likelihood estimator of S.

In a pioneer article, Renyi and Sulanke (1964) have shown that, if $d = 2$ and δS is of C_2-class, then

$$\mathrm{E}\left(\ell^2(S - \widehat{S}_n)\right) = \mathcal{O}(n^{-2/3}), \tag{2}$$

again this rate is optimal (KT (1993)). See also Mammen and Tsybakov (1995).

Note that $\widehat{S}_n$ possesses the three first qualities requested in Section 1, but not the fourth since $\widehat{S}_n \subset S$.

Extensions to estimation of an ε-convex support by the ε-convex hull of the sample is studied in Chevalier (1976).

3 Estimating the edge function

Suppose now that $d = 2$ and that S has the special form

$$S = \left\{z = (x, y),\ 0 \leq x \leq 1,\ 0 \leq y \leq g(x)\right\}$$

where g is a strictly positive continuous function on $[0, 1]$, called the *edge function* of S. It is assumed that μ has a density f_Z such that

$$0 < \min_S f_Z \leq \max_S f_Z < \infty. \tag{3}$$

Here, estimation of S reduces to estimation of g. For this purpose, Geffroy (1964) has proposed an histogram-type estimator : let k_n be a positive integer, set

$$I_{jn} = \Big[\frac{j-1}{k_n}, \frac{j}{k_n}\Big),\ 1 \leq j \leq k_n$$

where I_{jn} is open on the right, except $I_{k_n n}$ which is closed. The *Geffroy estimator* is defined by

$$g_n(x) = \max_{1 \leq i \leq n} \left[Y_i \, \mathbb{I}_{I_{jn}(x)}(X_i)\right],\ x \in [0, 1] \tag{4}$$

where $(X_i, Y_i) = Z_i$ and $I_{jn}(x)$ is the unique interval I_{jn} containing x.

The Geffroy estimator is easy to compute and we will see that it enjoys a good rate. However, it has a systematic bias and its shape is not satisfactory since g_n is not continuous.

A remedy consists in considering the modified estimator

$$\widetilde{g}_n = \gamma_n^{-1} g_n$$

where

$$\gamma_n = 1 - \frac{k_n}{n+1}\Big[1 - (1 - \frac{1}{k_n})^{n+1}\Big],$$

$\widetilde{g}_n$ is unbiased if g is constant and μ is uniform on S. Now, a linear interpolation of $\widetilde{g}_n$ provides a more suitable estimator of g.

Concerning g_n, Geffroy has obtained a sharp and amazing result. In order to state it we introduce a classical notation. Let $\varphi \ : A \to \mathbb{R}$ be a real function defined on some set A, then

$$\|\varphi\|_\infty = \sup_{x \in A} |\varphi(x)| \, .$$

Now, if $(k_n) \to \infty$, Geffroy has shown that the three following conditions are equivalent :

(i) $k_n = o\Big(\dfrac{n}{\ln n}\Big)$,

(ii) $\|g_n - g\|_\infty \to 0$ in probability,

(iii) $\|g_n - g\|_\infty \to 0$ almost surely (a.s.).

Some extensions of this result to a compact and a locally compact metric space appear in Bosq (1973).

We now study rate of convergence in a slightly different context. Let $g \ : \ \mathbb{R} \to \mathbb{R}$ be uniformly continuous, integrable and not degenerated. Put

$$G = \int_{-\infty}^{+\infty} |g(x)| \ \mathrm{d}x,$$

and

$$W(\delta) = \sup_{|x'-x|<\delta} |g(x') - g(x)| \, , \ \delta > 0.$$

Here S is the closed domain between the graph of g and the real axis. $Z_i = (X_i, Y_i)$, $1 \leq i \leq n$ is a sample of the uniform distribution on S with density $G^{-1}\mathbb{I}_S$. Note that S is, in general, not bounded.

The estimator of g is defined as

$$g_n(x) = \max_{1\le i\le n} Y_i \, \mathbb{I}_{I_{jn}(x)}(X_i)\mathbb{I}_{Y_i>0} + \min_{1\le i\le n} Y_i \, \mathbb{I}_{I_{jn}(x)}(X_i)\mathbb{I}_{Y_i<0}, \; x \in \mathbb{R},$$

where $I_{jn} = \Big[\dfrac{j-1}{k_n}, \dfrac{j}{k_n}\Big[$, $j \in \mathbb{Z}$, k_n is a strictly positive real number, and $I_{jn}(x)$ is the unique interval I_{jn} which contains x.

Finally G_M denotes an upper bound for G. The next statement provides a rate.

Theorem 1
1) If

$$k_n = \Big(c\,\frac{n}{\ln n}\Big)^{1/2} \tag{5}$$

where $0 < c < \frac{1}{2G_M}$ *then*

$$\|g_n - g\|_\infty \to 0 \;\; a.s. \tag{6}$$

2) If, in addition, we have

$$(A(\alpha)) \qquad w(\delta) \le \ell_\alpha \delta^\alpha, \;\; \delta > 0 \;\; (\ell_\alpha > 0, \; 0 < \alpha \le 1)$$

then,

$$\|g_n - g\|_\infty = \mathcal{O}\Big(\Big(\frac{\ln n}{n}\Big)^{\alpha/2}\Big) \;\; a.s. \tag{7}$$

and, for $0 < c \le \dfrac{2}{3G_M}$,

$$\mathrm{E}\;\|g_n - g\|_\infty = \mathcal{O}\Big(\Big(\frac{\ln n}{n}\Big)^{\alpha/2}\Big). \tag{8}$$

The proof shows that, almost surely for n large enough, we have

$$\|g_n - g\|_\infty \le 2\frac{2+\ell_\alpha}{c^{\alpha/2}}\Big(\frac{\ln n}{n}\Big)^{\alpha/2},$$

and that, for n large enough,

$$\mathrm{E}\;\|g_n - g\|_\infty \le 2\frac{2+\ell_\alpha}{c^{\alpha/2}}\Big(\frac{\ln n}{n}\Big)^{\alpha/2}.$$

Note that, if g is Lipschitz, the rate becomes $\left(\frac{\ln n}{n}\right)^{1/2}$; compare with (1).

We now study the integrated square error defined as

$$\Delta_n = \int_{-\infty}^{+\infty} \mathrm{E}\left(g_n(x) - g(x)\right)^2 \mathrm{d}x.$$

We have

Theorem 2
If $k_n = \left(c\,\dfrac{n}{\ln n}\right)^{1/2}$ *with* $0 < c \leq \frac{3}{2G_M}$, *if* $A(\alpha)$ *holds for some* $\alpha \in]0,1]$, *then*

1) if

$$|g(x)| \leq \frac{\gamma}{|x|^\delta}, \quad |x| \geq x_0 > 0, \quad (\gamma > 0,\ \delta > 1), \tag{9}$$

we have

$$\Delta_n = \mathcal{O}\Big(\Big(\frac{\ln n}{n}\Big)^{\frac{2\delta-1}{2\delta}}\Big), \tag{10}$$

2) if

$$|g(x)| \leq a\,\mathrm{e}^{-b|x|}, \quad x \in \mathbb{R} \ \ (a > 0,\ b > 0), \tag{11}$$

then

$$\Delta_n = \mathcal{O}\Big(\ln\ln n \cdot \frac{(\ln n)^2}{n}\Big). \tag{12}$$

The monotone case
If g is monotone on $[0,1]$, one may use special estimators. Actually, if g is decreasing, the graph of g is the set of maximal elements of S, then it is possible to construct a consistent estimator of g based on the maximal elements of $\{Z_1, \ldots, Z_n\}$. This has been done in Bosq (1973).

Korostelev et al. (1995) have shown that, due to the fact that g is monotone, the rate $\left(\frac{\ln n}{n}\right)^{1/2}$ is reached without regularity conditions concerning g. Moreover they have obtained the exact asymptotic error with respect to Hausdorff metric.

4 Using density estimators for estimating the support

If μ has a density f_Z, S is the closure of $\{f_Z > 0\}$. Then, it is natural to use nonparametric density estimators for estimating S. These estimators have the form

$$f_{Z,n}(z) = \frac{1}{n}\sum_{i=1}^{n} K_n(z, Z_i), \quad z \in \mathbb{R}^d$$

with various choice of K_n.

The *kernel estimator* corresponds to the choice of

$$K_n(z, z') = \frac{1}{h_n^d} K\Big(\frac{z'-z}{h_n}\Big) \ ; \quad (z, z') \in \mathbb{R}^{2d}$$

where K is a density and (h_n) a sequence of positive real numbers such that $\lim_{n\to\infty} h_n = 0$.

The *projection estimator* is associated with

$$K_n(z, z') = \sum_{j=1}^{\delta_n} e_j(z) e_j(z')$$

where (e_j) is an orthonormal system in $L^2(\mathbb{R}^d, \mathcal{B}_{\mathbb{R}^d}, \ell^d)$ and the truncation index (δ_n) is such that $\delta_n \to \infty$.

General studies of these estimators appear in Prakasa Rao (1983), Bosq and Lecoutre (1987) among others.

Smooth density

In the particular case where f_Z is a smooth density (for example a density of C_2 class) and S is compact, one may set

$$\widehat{S}_n = \Big\{ f_{Z,n} > \eta_n \Big\}$$

where $(\eta_n) \to 0\ (+)$. It is then easy to obtain good rates of convergence for $(\widehat{S}_n)$.

However this case is rather special since f_Z vanishes on δS.

Smooth estimation of the edge function

Consider estimation of g in the case where

$$S = \Big\{ (x, y) \ : \ 0 \le x \le 1,\ 0 \le y \le g(x) \Big\}.$$

If μ is uniform on S, X_i has the density

$$f(x) = \frac{g(x)}{G}, \ \ 0 \le x \le 1$$

where $G = \int_0^1 g(x)\,\mathrm{d}x$.

This suggests the following estimator of g :

$$\widehat{g}_n(x) = \int_0^1 g_n(u)\,\mathrm{d}u \cdot f_n(x), \quad 0 \le x \le 1$$

where g_n is the Geffroy estimator and f_n denotes a smooth estimator of f. Thus $\widehat{g}_n$ is a smooth estimator of g.

Consider, for example, the projection density estimator, with a data-driven truncation index $\widehat{\delta}_n$, defined in Bosq (2005). Then if $\sum_j (\int_0^1 g e_j) e_j$ converges uniformly and at an exponential rate, it can be shown that

$$\|\widehat{g}_n - g\|_\infty = \mathcal{O}\Big(\big(\frac{\ln n}{n}\big)^{1/2}\Big) \text{ a.s.}$$

It is the case if (e_j) is the trigonometrical basis and g is analytic and such that $g(0) = g(1)$. Thus, $\widehat{g}_n$ reaches the good rate and has the same regularity as g.

Estimation of a manifold

Suppose that $Z_i = (Y_i, X_i, \dots, X_{i-k})$ with

$$Y_i = \Phi(X_i, \dots, X_{i-k}) \quad (0 \le k \le d-2)$$

and that S is the manifold defined by

$$y = \Phi(x_1, \dots, x_{k+1}).$$

Then μ is orthogonal to ℓ^d and it can be shown that, under some regularity conditions, the kernel density estimator, say $f_n^{(K)}$, "explodes" in a neighborhood of S and "vanishes" elsewhere. This suggests to define an estimator Φ_n of Φ by

$$f_n^{(K)}(\Phi_n(x_0, \dots, x_k), x_0, \dots, x_k) = \max_{y \in \mathbb{R}} f_n^{(K)}(y, x_0, \dots, x_k).$$

Then, one has

$$\|\Phi_n - \Phi\|_\infty = \mathcal{O}\Big(\ln\ln n \big(\frac{\ln n}{n}\big)^{1/(k+1)}\Big) \text{ a.s.}$$

For details we refer to Bosq (1989), see also Bosq (1997).

5 Proofs

5.1 Proof of Theorem 1

Set

$$J_n = \Big\{ j \in \mathbb{Z},\ \inf_{I_{jn}} |g| \geq \frac{1}{k_n} \Big\}.$$

Since $g \not\equiv 0$, J_n is not void for n large enough. Moreover J_n is finite because g is integrable. Finally g has clearly a constant sign over I_{jn} if $j \in J_n$.

Now consider a covering of S by squares of the form $I_{jn} \times \big[\frac{\ell}{k_n}, \frac{\ell+1}{k_n}\big]$, $(j,\ell) \in \mathbb{Z}^2$. For $j \in J_n$ one sets

$$C_{jn} = I_{jn} \times \Big[\frac{\ell_{jn}}{k_n}, \frac{\ell_{jn}+1}{k_n}\Big]$$

where ℓ_{jn} is the greatest integer such that

$$\frac{\ell_{jn}}{k_n} \leq \inf_{I_{jn}} |g| \,.$$

To $(C_{jn},\ j \in J_n)$ one associates the event

$$E_n = (\forall\, j \in J_n)\ \ (\exists\, i \in \{1, \ldots, n\})\ \ :\ \ (X_i, Y_i) \in C_{jn}.$$

The elementary computations show that

$$\# J_n \leq G\, k_n^2$$

where $\#$ denotes cardinal, and that

$$P(E_n^c) \leq G k_n^2 \Big(1 - \frac{1}{G k_n^2}\Big)^n, \tag{13}$$

thus

$$P(E_n^c) \leq G k_n^2 \exp\Big(-\frac{n}{G k_n^2}\Big). \tag{14}$$

Now, if E_n is realized, each C_{jn} contains at least one point of the sample, then, for $g > 0$ on I_{jn},

$$\inf_{I_{jn}} g - \frac{2}{k_n} \leq g_n(x) \leq \sup_{I_{jn}} g, \quad x \in I_{jn},$$

and since

$$\sup_{I_{jn}} g - \inf_{I_{jn}} g \leq w\Big(\frac{1}{k_n}\Big)$$

it follows that

$$|g_n(x) - g(x)| \leq w\Big(\frac{1}{k_n}\Big) + \frac{2}{k_n} := \varphi(k_n), \ \ x \in I_{jn}.$$

The same bound holds for $g < 0$ on I_{jn}.

On the other hand, if $j \notin J_n$, we have

$$\inf_{I_{jn}} |g| < \frac{1}{k_n}$$

and since

$$|g_n(x)| \leq \sup_{I_{jn}} |g| < \frac{1}{k_n} + w\Big(\frac{1}{k_n}\Big)$$

one has

$$|g_n(x) - g(x)| < \varphi(k_n), \ \ x \in I_{jn}.$$

Now, using the relation

$$\|g_n - g\|_\infty = \|g_n - g\|_\infty \mathbb{I}_{E_n} + \|g_n - g\|_\infty \mathbb{I}_{E_n^c}$$

and the bounds obtained above, one obtains

$$\|g_n - g\|_\infty \leq \varphi(k_n) + 2\,\|g\|_\infty \mathbb{I}_{E_n^c}. \tag{15}$$

We are now in a position to obtain consistency and rate.

First, if $\varepsilon \in]0, 4\,\|g\|_\infty[$ then, for n large enough, $2\varphi(k_n) \leq \varepsilon$, therefore

$$P(\|g_n - g\|_\infty > \varepsilon) \leq P(E_n^c)$$

and from (14)

$$\begin{aligned} P(\|g_n - g\|_\infty > \varepsilon) &\leq G k_n^2 \exp\Big(-\frac{n}{G k_n^2}\Big) \\ &\leq \frac{Gc}{\ln n} \frac{1}{n^{\frac{1}{Gc} - 1}} = v_n. \end{aligned}$$

Condition $c < \frac{1}{2G_M}$ gives $\sum v_n < \infty$ and Borel-Cantelli lemma entails $\|g_n - g\|_\infty \to$ 0 a.s..

Now the choice $\varepsilon = 2\varphi(k_n)$ and Borel-Cantelli lemma imply

$$\|g_n - g\|_\infty = \mathcal{O}\big(\varphi(k_n)\big) \ \text{ a.s.}$$

and $A(\alpha)$ implies

$$\varphi(k_n) \leq \frac{2+\ell_\alpha}{k_n^\alpha} = \frac{2+\ell_\alpha}{c^{\alpha/2}}\Big(\frac{\ln n}{n}\Big)^{\alpha/2}$$

hence (7).

For mean convergence it suffices to note that (15) yields

$$\mathrm{E}\,\|g_n - g\|_\infty \;\leq\; \varphi(k_n) + 2\,\|g\|_\infty\, P(E_n^c) \;\leq\; \frac{2+\ell_\alpha}{c^{\alpha/2}}\Big(\frac{\ln n}{n}\Big)^{\alpha/2} + \frac{Gc}{\ln n}\frac{1}{n^{\frac{1}{Gc}-1}},$$

and $c \leq \dfrac{2}{3G_{_M}}$ implies $\dfrac{1}{Gc} - 1 \geq \dfrac{\alpha}{2}$ hence (8).

5.2 Proof of Theorem 2

Let j_n be a positive integer, we have

$$\begin{aligned}\Delta_n &= \sum_{|j|\leq j_n}\int_{I_{jn}}\Big(g_n(x)-g(x)\Big)^2\,\mathrm{d}x + \sum_{|j|>j_n}\int_{I_{jn}}\Big(g_n(x)-g(x)\Big)^2\,\mathrm{d}x\\ &:= A_n + B_n.\end{aligned}$$

Concerning A_n, using (15), and after some computations, one obtains

$$A_n \leq c^{1/2}(2j_n+1)\Big[2\frac{(2+\ell_\alpha)^2}{c^{3/2}}\Big(\frac{\ln n}{n}\Big)^{3/2} + 8\,\|g\|^2\,\frac{1}{n^{G_M c}}\Big]$$

and, since $c \leq \dfrac{3}{2G_{_M}}$,

$$A_n = \mathcal{O}\Big(j_n\Big(\frac{\ln n}{n}\Big)^{3/2}\Big).$$

On the other hand,

$$B_n \leq \frac{4}{k_n}\sum_{|j|>j_n}\sup_{I_{jn}}\left|g^2\right|$$

thus, for j_n large enough, and if (9) holds,

$$B_n \leq \frac{4\gamma^2}{k_n}\sum_{|j|>j_n}\frac{k_n^{2\delta}}{|j|^{2\delta}} = \mathcal{O}\Big(\Big(\frac{k_n}{j_n}\Big)^{2\delta-1}\Big)$$

choosing

$$j_n \simeq k_n^{\frac{\delta+1}{\delta}} = c^{\frac{\delta+1}{2\delta}}\Big(\frac{n}{\ln n}\Big)^{\frac{\delta+1}{2\delta}}$$

it follows that

$$\Delta_n = \mathcal{O}\Big(\Big(\frac{\ln n}{n}\Big)^{\frac{2\delta-1}{2\delta}}\Big).$$

Now, if (11) holds, we have

$$B_n \le \frac{4a^2}{k_n} \sum_{|j|>j_n} \mathrm{e}^{-b\frac{|j|}{k_n}}$$

hence

$$\begin{aligned} B_n &= \mathcal{O}\Big(\frac{1}{k_n}\frac{\mathrm{e}^{-b\frac{j_n}{k_n}}}{(1-\mathrm{e}^{-b/k_n})}\Big) \\ &= \mathcal{O}\Big(\mathrm{e}^{-b\frac{j_n}{k_n}}\Big) \end{aligned}$$

and the choice

$$j \simeq (n \ln n)^{1/2} \ln \ln n$$

gives

$$\Delta_n = \mathcal{O}\Big(\ln \ln n \frac{(\ln n)^2}{n}\Big).$$

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Geometrical Probabilities of Buffon Type in Bounded Lattices

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Abstract. Two extended examinations are summarized in form of an overview concerning geometrical probabilities in bounded lattices; all proofs of the here presented theorems can be found in [1]. We admit arbitrarily long needles and present the geometric probabilities p_k of exactly $k = 0, 1, ..., m, m{+}1$ intersections of needles $\mathcal{N}$ with the lattice $\mathcal{L}$ of equidistant parallel line segments of length d and a units apart consisting of m elementary cells — thus it is possible to generalize the results of Kendall and Moran in [5].

Introduction

Geometrical probabilities of various test objects and lattices are considered for instance in [2] to [4]. In this paper we shrink the lattice to a so called bounded one with finite expansion and examine the probability of non-small needles to intersect exactly k line segments of the lattice. We examine the case of needles which fall *onto* the convex hull of the lattice (**problem 1**) as well as the case of needles which fall completely *into* the convex hull of it (**problem 2**).

Definition 0.1 (Lattice)**.** *A plane* bounded lattice $\mathcal{L} := \mathcal{L}^m_{a,d}$ *is defined by a set of* $m \in \mathbb{N}$ *elementary rectangular cells containing* $m + 1$ *parallel line segments* $\mathcal{H}$*, which are of length* $d := 2r > 0$ *and* $a > 0$ *units apart:*

$$\mathcal{L}^m_{a,d} := \left\{ (x, y) \in \mathbb{R}^2 \mid -r < x < r,\, y = H_\nu := \nu a,\, \nu = 0, 1, ..., m \right\}.$$

The vertices of the ν*th line segment* $\mathcal{H}_\nu$ *are labeld* $\mathrm{E}^-_\nu := (-r, \nu a)$ *and* $\mathrm{E}^+_\nu := (r, \nu a)$ *respectively. Finally we will denote by* L_ν *the length of the diagonal* $\overline{\mathrm{E}^-_0 \mathrm{E}^+_\nu}$ *which is equal to* $\overline{\mathrm{E}^+_0 \mathrm{E}^-_\nu}$*, see figure 1.*

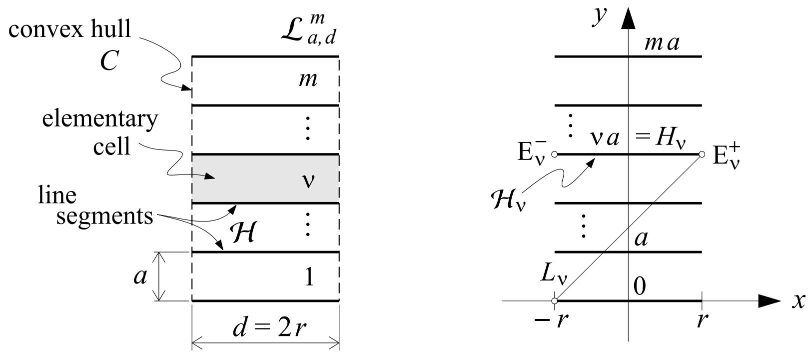

Fig. 1: Bounded lattice $\mathcal{L}^m_{a,d}$ and embedding into cartesian coordinates

Definition 0.2 (Needle)**.** *Let C be the convex hull of the lattice $\mathcal{L}$ and $\mathcal{N}$ a straight needle, i.e. line segment, of length $l > 0$ placed by random in the plane which has at least one point in common with the convex hull C, i.e. $\mathcal{N} \cap C \neq \emptyset$.*

Based on this geometrical properties we formulate the following two scenarios of geometrical probabilities:

Problem 1. (Needle on Open Lattice.) *What are the geometrical probabilities p°_k that a needle $\mathcal{N}$ intersect exactly $0 \le k \le m+1$ line segment of the lattice $\mathcal{L}$?*

Problem 2. (Needle in Closed Lattice.) *What are the geometrical probabilities $p^\bullet_k$ that a needle $\mathcal{N}$ intersect exactly $0 \le k \le m-1$ line segment of the lattice $\mathcal{L}$ under the restriction that $\mathcal{N}$ has to fall completely into the convex hull C, i.e. $\mathcal{N} \subset C$?*

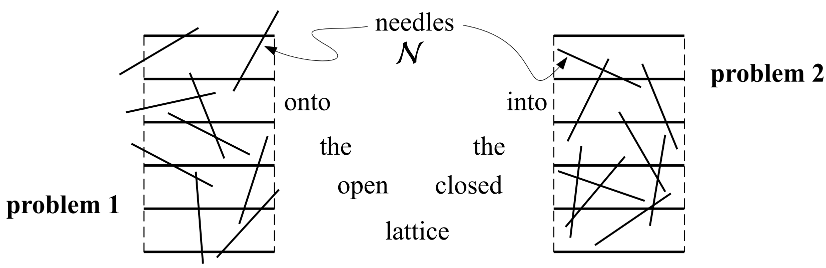

Fig. 2: Needles on an open and within a closed lattice

To gain in unity dimensionless parameters we introduce the following definition. An additional advantage is that the *three* sizes l of the needle $\mathcal{N}$ and a and d of the lattice $\mathcal{L}^m_{a,d}$ are expressed by only *two* ones.

Definition 0.3 (Parameters). *Let $\mathcal{L}_{a,d}^m$ the lattice according to definition 0.1 and $\mathcal{N}$ a needle with length $l > 0$ by definition 0.2. Then we introduce the parameters $\lambda := \frac{a}{d}$ (lattice) and $\alpha := \frac{a}{l}$ (needle).*

A needle $\mathcal{N}$ is represented in terms of extended polar coordinates. The needle $\mathcal{N}$ lies on the line $\mathcal{G} : x\cos\theta + y\sin\theta = \rho$ where ρ is the distance of the line from the origin and θ is the angle which the perpendicular makes with the x-axis. Then the bases point B has the polar coordinates $(\theta,\, \rho)$. The left vertex V of $\mathcal{N}$ in the sense of looking from the origin to the line $\mathcal{G}$ then is ζ units apart to point B. So the needle $\mathcal{N}$ is completly described in the coordinate system $(\theta,\, \rho,\, \zeta)$. In [6] we find the element dH of the invariant measure μ for such representation of a line segment in the plane by $dH = c\, d\zeta \wedge d\rho \wedge d\theta$ where c is a constant which will be 1 in the following.

Definition 0.4 (Set of Needles). *According to definition 0.2 we call $N^\circ := \{\mathcal{N} \,|\, \mathcal{N} \cap C \neq \emptyset\}$ and $N^\bullet := \{\mathcal{N} \,|\, \mathcal{N} \subset C\}$ the set of all needles $\mathcal{N}$ which have at least one point in common with C and which fall completely into C respectively. Similarly we call $N_k^\circ \subset N^\circ$ and $N_k^\bullet \subset N^\bullet$ the set of needles intersecting the lattice $\mathcal{L}_{a,d}^m$ exactly k times under the condition $\mathcal{N} \cap C \neq \emptyset$ and $\mathcal{N} \subset C$ respectively.*

So using the formula of Stoka the geometrical probabilities in problem 1 and problem 2 are the ratios $p_k^\circ = \mu(N_k^\circ)/\mu(N^\circ)$ and $p_k^\bullet = \mu(N_k^\bullet)/\mu(N^\bullet)$ respectively which will be expressed as functions of the parameters α, λ and the number m of elementary cells in the lattice $\mathcal{L}_{a,d}^m$.

1 The Open Lattice

The measure of the set N° in definition 0.4 is according to [6] given by $\pi A + l\,U$ where A is the area and U is the perimeter of the convex set C, i.e. with C as a rectangular of width d and height ma we have

$$\mu(N^\circ) \;=\; \pi A + l\,U \;=\; 2ml^2 \tfrac{\alpha}{\lambda}\big(\tfrac{\pi}{2}\alpha + \lambda + 1\big)\,. \tag{1}$$

On the open lattice the number of intersections k can range between 0 and $m+1$. We will calculate for the intersection numbers $k = 0$, $1 \leq k \leq m$ and $k = m+1$ different formulas. Furthermore it is necessary to introduce within this number of cuts several case distinctions. This is a long examination which can be studied in full scope in [1]. Here we condense the discussion by the following paragraph.

1.1 The case distinctions

To begin with $k = 0$: We distinguish whether the length l of the needle is shorter than the height a of an elementary cell and call this case (**0.1**). If the needle is longer than the height a but shorter than the length L_1 we will have the case (**0.2**). A needle longer than the diagonal of an elementary cell is considered in the last case (**0.3**). For $k = 1$ we have to consider furthermore the relationship between l and the length $2a$ and L_2, i.e. the height and the length of the diagonal of two elementary cells. Then there are six case distinctions called (**I.1**) to (**I.6**). For the numbers $1 < k < m+1$ of cuts we combine the relationship between the length l and the heights κa, $\kappa = k-1, k, k+1$, by

$$\begin{array}{lllll} (\mathbf{0.}) : & & l < (k-1)a, & \text{no } k \text{ intersections possible,} \\ (\mathbf{1.}) : & (k-1)a \le & l < ka, & k, \text{ but not } k+1 \text{ intersections possible,} \\ (\mathbf{2.}) : & ka \le & l < (k+1)a, & k+1 \text{ intersections possible,} \\ (\mathbf{3.}) : & (k+1)a \le & l, & \text{more than } k+1 \text{ intersections possible} \end{array}$$

with the relationship between the length l and the lengths of the diagonals L_κ, $\kappa = k-1, k, k+1$:

$$\begin{array}{llll} (\mathbf{C.1}) : & & l < L_{k-1} & \text{with } \mathbf{C} = \mathbf{1}, \mathbf{2}, \mathbf{3}, \text{ and } (k-1)a \le l, \\ (\mathbf{C.2}) : & L_{k-1} \le & l < L_k & \text{with } \mathbf{C} = \mathbf{1}, \mathbf{2}, \mathbf{3}, \\ (\mathbf{C.3}) : & L_k \le & l < L_{k+1} & \text{with } \mathbf{C} = \mathbf{2}, \mathbf{3}, \\ (\mathbf{C.4}) : & L_{k+1} \le & l & \text{with } \mathbf{C} = \mathbf{3}. \end{array}$$

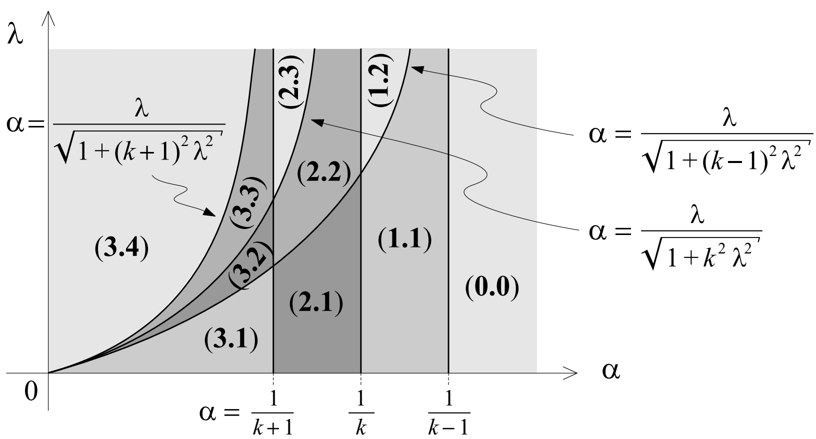

Fig. 3: Case distinctions for $1 < k < m+1$

Finally for $k = m + 1$ we have once more three subcases (**L.1**) to (**L.3**): $0 < l < ma$, $ma \leq l < L_m$, and $L_m \leq l$. The characters (**0**), (**I**), (**C**), and (**L**) of the subcases should remember to the numbers of intersections, i.e. no cuts at all, one cut, general cuts, and intersecting the whole lattice respectively. In [1] it is discussed in detail that there are no further case distinctions necessary.

Annotation 1.1 (Subcases for the Open Lattice)**.** *Using the parameters α and λ of definition 0.3 all distinctions of subcases can be summarized by the following inequality: $\alpha_1 < \alpha \leq \alpha_2$ with $\alpha_1 = \alpha_1(k, \lambda)$ and $\alpha_2 = \alpha_2(k, \lambda)$ according to the following table and the short form $w_k := \frac{\lambda}{\sqrt{1+k^2\lambda^2}}$.*

$k = 0$:

	α_1	α_2		α_1	α_2		α_1	α_2
(**0.1**)	1	∞	(**0.2**)	w_1	1	(**0.3**)	0	w_1

$k = 1$:

	α_1	α_2		α_1	α_2		α_1	α_2
(**I.1**)	1	∞	(**I.2**)	$\max(\frac{1}{2}, w_1)$	1	(**I.3**)	$\frac{1}{2}$	w_1
(**I.4**)	w_1	$\frac{1}{2}$	(**I.5**)	w_2	$\min(w_1, \frac{1}{2})$	(**I.6**)	0	w_2

$1 < k < m + 1$: see figure 3

	α_1	α_2		α_1	α_2
(**0.0**)	$\frac{1}{k-1}$	∞	(**1.1**)	$\max(\frac{1}{k}, w_{k-1})$	$\frac{1}{k-1}$
(**1.2**)	$\frac{1}{k}$	w_{k-1}	(**2.1**)	$\max(\frac{1}{k+1}, w_{k-1})$	$\frac{1}{k}$
(**2.2**)	$\max(\frac{1}{k+1}, w_k)$	$\min(w_{k-1}, \frac{1}{k})$	(**2.3**)	$\frac{1}{k+1}$	w_k
(**3.2**)	w_k	$\min(w_{k-1}, \frac{1}{k+1})$	(**3.1**)	w_{k-1}	$\frac{1}{k+1}$
(**3.3**)	w_{k+1}	$\min(w_k, \frac{1}{k+1})$	(**3.4**)	0	w_{k+1}

$k = m + 1$:

	α_1	α_2		α_1	α_2		α_1	α_2
(**L.1**)	$\frac{1}{m}$	∞	(**L.2**)	w_m	$\frac{1}{m}$	(**L.3**)	0	w_m

1.2 The probabilities for a needle on the open lattice

In this section we will present the following main result about arbitrarily long and randomly placed needles on an open lattice which is the solution of problem 1. The proof can be found in [1].

Theorem 1.2. *The probability* $p_k^\circ = p^\circ(\alpha,\lambda,m,k)$ *that a random needle* $\mathcal{N}$ *according to* **problem 1** *has exactly* k *intersections with the lattice* $\mathcal{L}$ *is expressed by:*

case (**0**) $k=0$: $\quad p^\circ(\alpha,\lambda,m,0) = \gamma\cdot\left(\frac{\pi}{2}\,\alpha - 1 + H^{(\mathbf{0.c})}\right) \qquad (2)$

with the expressions

$(\mathbf{0.1})$: $H^{(\mathbf{0.1})} = \lambda$, $(\mathbf{0.2})$: $H^{(\mathbf{0.2})} = \lambda\cdot\alpha + \Phi_1$, $(\mathbf{0.3})$: $H^{(\mathbf{0.3})} = \Psi_1$,

case (**C**) $1 < k < m+1$:

$$p^\circ(\alpha,\lambda,m,k) = \frac{2}{m}\,\gamma\cdot\left[\lambda\,A_k^{(\mathbf{C.c})} + \delta_1\,\Omega_{k-1}^{(\mathbf{C.c})} - 2\delta_{\frac{1}{2}}\,\Omega_k^{(\mathbf{C.c})} + \delta_0\,\Omega_{k+1}^{(\mathbf{C.c})}\right] \qquad (3)$$

with the following expressions

$(\mathbf{C.c})$	$A_k^{(\mathbf{C.c})}$	$\Omega_{k-1}^{(\mathbf{C.c})}$	$\Omega_k^{(\mathbf{C.c})}$	$\Omega_{k+1}^{(\mathbf{C.c})}$
$(\mathbf{0.0})$	0	0	0	0
$(\mathbf{1.1})$	$-\delta_1\cdot(k-1)(1-(k-1)\alpha)$	Φ_{k-1}	0	0
$(\mathbf{1.2})$	$-\delta_1\cdot(k-1)$	Ψ_{k-1}	0	0
$(\mathbf{2.1})$	$(1-k\alpha) - \delta_1\cdot(k-1)\,\alpha + \delta_0\cdot(k+1)(1-k\alpha)$	Φ_{k-1}	Φ_k	0
$(\mathbf{3.1})$	$(1-k\alpha) - \delta_1\cdot(k-1)\,\alpha + \delta_0\cdot(k+1)\,\alpha$	Φ_{k-1}	Φ_k	Φ_{k+1}
$(\mathbf{2.2})$	$(1-k\alpha) - \delta_1\cdot(k-1)k\alpha + \delta_0\cdot(k+1)(1-k\alpha)$	Ψ_{k-1}	Φ_k	0
$(\mathbf{3.2})$	$(1-k\alpha) - \delta_1\cdot(k-1)k\alpha + \delta_0\cdot(k+1)\,\alpha$	Ψ_{k-1}	Φ_k	Φ_{k+1}
$(\mathbf{2.3})$	$1 + \delta_0\cdot(k+1)$	Ψ_{k-1}	Ψ_k	0
$(\mathbf{3.3})$	$1 + \delta_0\cdot(k+1)^2\alpha$	Ψ_{k-1}	Ψ_k	Φ_{k+1}
$(\mathbf{3.4})$	1	Ψ_{k-1}	Ψ_k	Ψ_{k+1}

and $\alpha := \frac{a}{l}$, $\lambda := \frac{a}{d}$, $\delta_i := i + \frac{m}{2} - \frac{k}{2}$, $i = 1, \frac{1}{2}, 0$,

$$\gamma := \left(\tfrac{1}{m} + \tfrac{\pi}{2}\,\alpha + \lambda\right)^{-1} \quad \textit{and} \quad \Phi_0 := \Psi_0 := 1,$$

$$\Phi_\kappa := \sqrt{1-\kappa^2\alpha^2} - \kappa\alpha\,\arccos(\kappa\alpha) - \kappa^2\alpha\lambda\,\ln(\kappa\alpha),$$

$$\Psi_\kappa := \sqrt{1+\kappa^2\lambda^2} - \kappa\alpha\,\arctan\frac{1}{\kappa\lambda} - \kappa^2\alpha\lambda\,\ln\frac{\kappa\lambda}{\sqrt{1+\kappa^2\lambda^2}},\ \kappa > 0.$$

The probabilities of the cases (**I**) $k=1$ *and* (**L**) $k=m+1$ *are receivable from case* (**C**) *by the following identifications:*

case (**I**) $k=1$: (**I.1**) $\equiv$ (**1.2**), (**I.2**) $\equiv$ (**2.2**), (**I.3**) $\equiv$ (**2.3**), (**I.4**) $\equiv$ (**3.2**), (**I.5**) $\equiv$ (**3.3**), (**I.6**) $\equiv$ (**3.4**),

case (**L**) $k=m+1$: (**L.1**) $\equiv$ (**0.0**), (**L.2**) $\equiv$ (**1.1**), (**L.3**) $\equiv$ (**1.2**).

1.3 Mean number of cuts

The function p_k° can also be read as the density of the random variable

$$X^\circ : C \to \mathbb{N}_0,\ \mathcal{N} \mapsto X(\mathcal{N}) := \#\{\text{ intersections between } \mathcal{N} \text{ and } \mathcal{L}_{a,d}^m\}.$$

So it is naturally to ask for the expectation value of X° which gives us the mean number of cuts between a randomly placed needle on the convex hull C and the lattice $\mathcal{L}_{a,d}^m$. For this we have to evaluate the sum over $k\,p^\circ(\alpha, \lambda, m, k)$ from $k = 0$ to $m + 1$. Due to the large number of case distinctions this seems to be a cumbersome work. But the directed graph in figure 4 brings some help. There is a strict way of changing the subcase distinctions while increasing the number k of intersections: there are 26 sequences. (Note that for $k = m$ the subcases (**2.1**), (**3.1**) and (**2.2**), (**3.2**) and (**2.3**), (**3.3**), (**3.4**) can be unified.) A detailed examination of this sum can be find in [1] which yields the following result:

Theorem 1.3. *According to problem 1 the mean number of intersections between a randomly placed needle $\mathcal{N}$ on the open lattice $\mathcal{L}$ is*

$$E(X^\circ)(\alpha, \lambda, m) = \sum_{k=0}^{m+1} k\,p^\circ(\alpha, \lambda, m, k) = \frac{1 + \frac{1}{m}}{\frac{\pi}{2}\alpha + \lambda + \frac{1}{m}}.$$

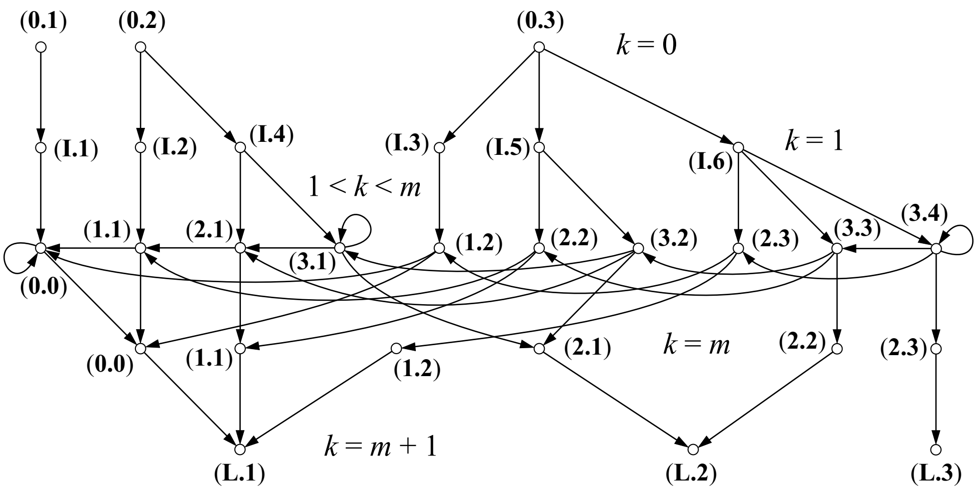

Fig. 4: Changing of case distinctions while increasing the number of intersections

2 The Closed Lattice

In the closed lattice the number of intersections k can only range between 0 and $m-1$. Nevertheless the number of case distinctions is also very large and therefore we will give the geometric probability $p_k^\bullet$ in form of an assembly system, this means that we will not write down all formulas explicitly. But all parts are developed to assemble the probability $p_k^\bullet$ due to the case distinction the needle suffices.

2.1 The discussion of the case distinctions

Definition 2.1. *We call C_κ the convex hull of the lattice $\mathcal{L}_{a,d}^{\kappa}$ with $1 \leq \kappa \leq k+1 \leq m$. For $\mathcal{N}$ as a needle according to problem 2 we define $B_\kappa := \{\mathcal{N} \,|\, \mathcal{N} \subset C_\kappa\}$ as the set of all needles in a box with κ cells.*

With figure 5 we introduce the following case distinctions for needles in relation to a closed box.

Annotation 2.2 (Subcases for the Closed Lattice)**.** *With the parameters of definition 0.3 a needle $\mathcal{N} \subset C_\kappa$ in a closed box can be classified by*

$$(\kappa.\mathbf{1}):\ \max(\lambda, \tfrac{1}{\kappa}) < \alpha\,, \qquad (\kappa.\mathbf{2}):\ \tfrac{1}{\kappa} < \alpha \leq \lambda\,,$$
$$(\kappa.\mathbf{3}):\ \lambda < \alpha \leq \tfrac{1}{\kappa}\,, \qquad (\kappa.\mathbf{4}):\ \tfrac{\lambda}{\sqrt{1+\kappa^2\lambda^2}} < \alpha \leq \min(\lambda, \tfrac{1}{\kappa})\,.$$

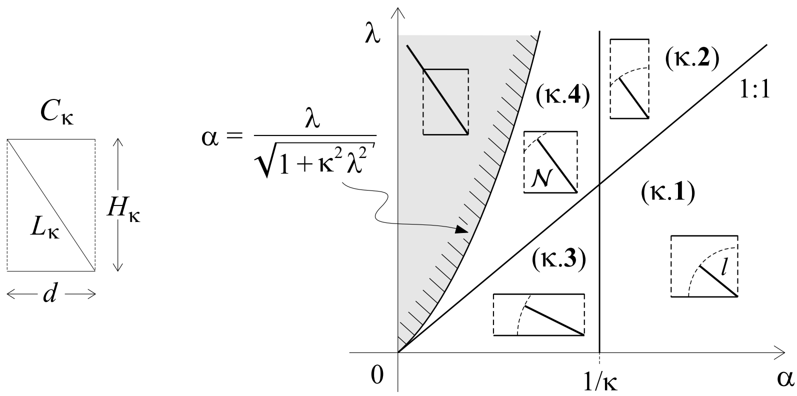

Fig. 5: Case distinctions for a the relations of a needle inside a box C_κ

We will see that all probabilities $p_k^\bullet$ can be assembled by the measures of the sets B_κ with $1 \leq \kappa \leq k+1 \leq m$.

Lemma 2.3. *The measure $\mu(B_\kappa)$ according to the case distinctions in notation 2.2 is equal to*

$(\kappa.\mathbf{1})$: $\mu_{(\kappa.\mathbf{1})} = \Lambda\left(\pi\kappa\alpha - 2\kappa\lambda - 2 + \frac{\lambda}{\alpha}\right)$, $(\kappa.\mathbf{2})$: $\mu_{(\kappa.\mathbf{2})} = \Lambda\left(\Xi_\kappa - 2\kappa\lambda\right)$,
$(\kappa.\mathbf{3})$: $\mu_{(\kappa.\mathbf{3})} = \Lambda\left(\Upsilon_\kappa + \pi\kappa\alpha - 2\right)$, $(\kappa.\mathbf{4})$: $\mu_{(\kappa.\mathbf{4})} = \Lambda\left(\Xi_\kappa + \Upsilon_\kappa - \frac{\lambda}{\alpha}\right)$.

$$\begin{aligned} \textit{with} \quad \Xi_\kappa &:= 2\kappa\lambda\sqrt{1-(\tfrac{\alpha}{\lambda})^2} + 2\kappa\alpha\arcsin(\tfrac{\alpha}{\lambda}) - \tfrac{\alpha}{\lambda}\,, \\ \Upsilon_\kappa &:= 2\sqrt{1-(\kappa\alpha)^2} - 2\kappa\alpha\arccos(\kappa\alpha) - \kappa^2\alpha\lambda\,, \quad \Lambda := l^2\,\tfrac{\alpha}{\lambda}\,. \end{aligned}$$

This result gives us directly the following probability for no intersections of needles in the closed lattice $\mathcal{L}^m_{a,b}$, i.e. for needles which are placed in boxes of the heights H_1 and H_m respectively: $p_0^\bullet = \mu(N_0^\bullet)/\mu(N^\bullet) = m \cdot \mu(B_1)/\mu(B_m)$ according to all possible combinations of case distinctions for needles in the sets B_1 *and* B_m. So we have the

Corollary 2.4. *The solution of* **problem 2** *for a needle with lenght $l < L_1$ and for $k = 0$ is*

case $(\mathbf{1.1})$:	$p_0^\bullet = m\,\frac{\mu_{(1.\mathbf{1})}}{\mu_{(m.\mathbf{1})}}$,	case $(\mathbf{1.2})$:	$p_0^\bullet = m\,\frac{\mu_{(1.\mathbf{2})}}{\mu_{(m.\mathbf{2})}}$,
case $(\mathbf{1.3})\cap(\mathbf{m.1})$:	$p_0^\bullet = m\,\frac{\mu_{(1.\mathbf{3})}}{\mu_{(m.\mathbf{1})}}$,	case $(\mathbf{1.4})\cap(\mathbf{m.2})$:	$p_0^\bullet = m\,\frac{\mu_{(1.\mathbf{4})}}{\mu_{(m.\mathbf{2})}}$,
case $(\mathbf{m.3})$:	$p_0^\bullet = m\,\frac{\mu_{(1.\mathbf{3})}}{\mu_{(m.\mathbf{3})}}$,	case $(\mathbf{m.4})$:	$p_0^\bullet = m\,\frac{\mu_{(1.\mathbf{4})}}{\mu_{(m.\mathbf{4})}}$.

For needles with $l > L_1$, i.e. the needle is longer than the diagonal of an elementary cell, it is of course $p_0^\bullet = 0$.

For a general number of intersections k it is not very convenient to have in mind whether a needle $\mathcal{N}$ fits into a box of height $(k+1)a$ or not, i.e. whether $\mathcal{N}$ is suitable for one the case distinctions or $B_{k+1} = \emptyset$. So we introduce the following function in

Definition 2.5. *With the measures of the set B_κ to lemma 2.3 we define the function*

$$b_\kappa := b(\alpha,\lambda,l,\kappa) := \begin{cases} \mu_{(\kappa.\mathbf{i})}, & \mathcal{N} \textit{ acc. to case } (\kappa.\mathbf{i})\,, \\ & \textit{with } \mathbf{i} = \mathbf{1},\mathbf{2},\mathbf{3} \textit{ or } \mathbf{4}\,, \\ 0, & \textit{else, i.e. } l > L_{\kappa+1}\,. \end{cases} \tag{4}$$

In the sense of the assembly concept to corollary 2.4 we gain the following

Lemma 2.6. *The solution of* **problem 2** *in form of a recurrence in* $p_k^\bullet$ *is*

$$p_k^\bullet = (m-k)\cdot\left[\, q_{k+1} - \sum_{\nu=0}^{k-1} \frac{k+1-\nu}{m-\nu}\, p_\nu^\bullet \,\right] \tag{5}$$

with the quotient $q_{k+1} := \frac{b_{k+1}}{b_m}$ *and the probability* $p_0^\bullet = m\,\frac{b_1}{b_m}$ *for the beginning of the recursion formula.*

The numerator in q_{k+1} is just a measure and independent of the probabilities $p_\nu^\bullet$, $0 \le \nu \le k-1$, so (5) is actually a recurrence. It is now easy to gain the following theorem by induction, see once more [1].

Theorem 2.7. *The solution of equation (5) is* $p_k^\bullet = p_k^\bullet(\alpha,\lambda,m,k) = \frac{m-k}{b_m}\cdot(b_{k-1} - 2\,b_k + b_{k+1})$ *with* $k = 0, 1, ..., m-1$ *and* $b_{-1} := b_0 := 0$.

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NO-CONVEX ENERGY IN MICROSTRUCTURAL CHARACTERIZATION OF SOLIDS

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Abstract. We represent a new characterization of the equilibrium problems on solids, particularly like crystalline type. Today great interests appear regarding the modelization of new materials like shape memory alloy, polymers and magneto-strictive materials. Starting from energetic considerations we deduce that energetic approach of the equilibrium problem fails because the hypothesis on the deformation energy convexity does not appear physically exact. Since solid-solid phase transitions models can be applied to materials behavior, it is possible to use the poly-convex stored energy functional on the modelization of solids response at microstructural scale. We regard a simple mono-dimensional solid undergoing to axial, bending and warping actions. Putting the equilibrium conditions we find approximate solutions by minimizing sequence such that, a good agreement between numerical and experimental results appear.

1. Introduction

To better understand the importance of the modelization in mechanics of materials we refer to new materials and their mechanical property that appear in many fields of engineering.

Under this framework, the microstructural characterization appears as advanced tool to formulate equilibrium and stability of materials behaviour models.

This approach derived from solid-solid phase transition theory represents a modelization tool such that typical non linear important phenomena can be handled within the finite elasticity framework.

In this paper after the classical questions we focalize the attention on the variational approach to the equilibrium problem in monodimensionale case, starting by the consideration that the stored deformation energy has poly-convex form.

Since convexity fails, the direct methods, in the calculus of variation, cannot be applicable and approximate solutions must be constructed by minimizing sequence to the total energy functional.

This last formulation, according with experimental results, permits to build models with deformation concentration and elastic and plastic phases co-existence typical of almost all new and old materials.

2. Remark of Finite and Linear Elasticity

Let us B a continuum hyperelastic body and let Ω_B the configuration reference in the Euclidean space $\Re^3$. The body boundary $\partial\Omega_B = \partial\Omega_B\square \cup \partial\Omega_B\square$ is the union of the free $\partial\Omega_B\square$ and restraint part $\partial\Omega_B\square$.

The deformation function $f : \Omega_B \rightarrow \Re^3$ is the one to one map and let us u the displacement field over Ω_B and F the deformation gradient. In linear elasticit the following kinematics equations hold [1]

$$\mathbf{E} = \tfrac{1}{2}(\nabla u + \nabla u^{\mathbf{T}}) \in \mathrm{Sym}^{+}$$
$$\square = \tfrac{1}{2}(\nabla u - \nabla u^{\mathbf{T}}) \in \mathrm{Skew}$$

Let T the Cauchy stress tensor and let f and s□ respectively , mass and surface forces, then the balance forces follow.

$$\begin{array}{ll} \mathrm{div}\,\mathbf{T} + \mathbf{f} = \square & \textbf{in } \Omega_B \\ \mathbf{s} = \mathbf{Tn} & \textbf{in } \partial\Omega_B \\ \mathbf{T} = \mathbf{T}^{\mathbf{T}} & \textbf{in } \Omega_B \\ \mathbf{T}^{k} = \det\mathbf{F}(\mathbf{TF}^{-\mathbf{T}}) & \textbf{in } \Omega_B \end{array}$$

Let $\mathbb{C} \in \mathrm{Sym}$ the fourth-order elasticity tensor, then the deformation store energy and the total energy functional in Ω_B take the forms

$$W = \int_B \tfrac{1}{2}(\mathbb{C}\mathbf{E} \cdot \mathbf{E})$$
$$\mathcal{E} = \int_B W + \mathcal{L}$$

where $\mathcal{L}$ represent the functional of the external work.

In linear elasticity , one important consequence of the Kirchhoff's theore resulting how :

- $\mathbb{C} \geq 0$
- only one solutions exist to the boundary value problem

In the finite elasticity framework, recalling the polar decomposition theo $\forall \mathbf{F} \in \mathrm{Lin}^{+}\ \exists \mathbf{R} \in \mathrm{Orth} \rightarrow \mathbf{F} = \mathbf{RU} = \mathbf{VR}$ with $\mathbf{U},\mathbf{V} \in \mathrm{Lin}^{+}$, we keep a fine deformation measure as

$$\mathbf{U}^2 = \mathbf{C} = \mathbf{F}^{\mathbf{T}}\mathbf{F}$$
$$\mathbf{V}^2 = \mathcal{B} = \mathbf{FF}^{\mathbf{T}}$$

where **C** and $\mathcal{B}$ are respectively, left and right Cauchy deformation tensors.

From energetic evaluations putting the arguments of W as

$$W(\cdot\,, t) = W(\mathbf{F}(\cdot\,, t))$$

we define the hyperelasticity material when the Piola-Kirchhoff stress tensor can be derived by means

$$\mathbf{T}^k = \partial W(\mathbf{F})/\partial \mathbf{F} = 2\mathbf{F}[\partial W(\mathbf{C})/\partial \mathbf{C}]$$

and so, the constitutive law has the final form

$$\mathbf{T} = \alpha_0 \mathbf{I} + \alpha_1 B + \alpha_2 B^2$$

• Classical questions in finite elasticity: the minimum problem

Recaling the energetic balance problem in the form

$$\mathcal{E}(\mathbf{E}) = \int_B W(\mathbf{E}(u)) + \mathcal{L}(x, \mathbf{E}(x))$$

in the admissible set

$$\{u \in \mathbf{H}^{1,p}(\Omega, \Re^3) \mid \mathcal{E}(\mathbf{E}) = \min + \text{b.c.}\}$$

For instance, considering a 1-D case [2], W, is a Carathèodory function in the set $[0, 1] \times (0, +\infty)$, coercive and $u \in \mathbf{C}^1(0,1)$. Let v a function test, in this conditions for any perturbation, equilibrium and stability are formulated as

$$W(u + \varepsilon v)_{\varepsilon|\varepsilon=0} = 0$$
$$W(u + \varepsilon v)_{\varepsilon\varepsilon|\varepsilon=0} \geq 0$$

A direct consequence of the minimum problem appear in the relationship

$$\text{convexity } W \Rightarrow \mathbf{T} \text{ monotonic } \Rightarrow \partial^2 W/\partial \mathbf{E}^2 \geq 0$$

• Variational approach: abstract formulation and direct methods

Let us Ω a topological space and $f:\Omega \rightarrow \Re$ a one to one map. According to Kolmogorov & Fomin [3] the function f is:

1- lower semi-continuous if $\forall m$ the set $\{u \in \Omega:\ f(u) \leq m\}$ is closed (strong hypothesis).

2- coercive if $\forall m\ \exists K_m \subset \Omega$, compact set, such that $\{u \in \Omega:\ f(u) \leq m\} \subset K_m$.

When Ω is a metric space, then a sequential characterization will be proposed and the formulation assume the form $\forall \{u_n\} \rightarrow u \in \Omega$, f is sequential lower semi-continuous and so, $f(u) \leq \liminf_{n\rightarrow\infty} f(u_n)$.

Sequential coercive if $\forall \{u_n\} \Rightarrow f(u_n) \leq m\ \exists f(u_{nk}) \rightarrow u \Rightarrow f(u) \leq m$.

An important theorem follows:

Theorem (Tonelli) Let f $\Omega\to\Re$ an one to one map. If the function f have the property: 1 sequential lower semi-continuous, 2 sequential coercive, then $\exists \min f \in\Omega$, namely $f(u^*) \le \liminf_{n\to\infty} f(u_{nk}) \le \inf f(u)$: $u\in\Omega$.

Under these conditions only one solution exists, in Hadamard lemma sense, and so good solution is verified. But, in finite elasticity no-homogeneous solution represent the real behaviour of materials (see swelling, barrel buckling and other non linear phenomena).

About variational elasticity and convergence problems the poly-convex stored energy of J. Ball [4] represents a good tool to the question treatment. In fact, it represents the stored energy as function of many variables, particularly the final form

$$W(x, \mathbf{F}) = W^*(x, \mathbf{F}, \operatorname{cof}\mathbf{F}, \det\mathbf{F})$$

Now, to solve our questions, let us consider an homogeneous continuum iper-elastic solid undergoing to deformation field u: $\Omega_B \to \Re^n$ and let $\mathbf{F} = \nabla u$ the deformation gradient. Putting the equilibrium problem in the form

$$\min \mathcal{E}(u)_{u\in\mathcal{A}} = \int_\Omega W(u)\, d\Omega$$

$$\mathcal{A}=\{u\in \mathbf{H}^{n,p}(\Omega), \operatorname{cof}\mathbf{F}, \det\mathbf{F}>0\}$$

where $\mathcal{A}$ **represent the admissible deformation set (Sobolev space).**

For instance, considering a typical poly-convex energy, (Bal-energy) we observe that the multi-well energy hypotesis involve equilibrium as phase mixture at fine scale.

Finally a variational formulation is build over the follow:

Theorem: let Ω a configuration of an elastic, homogeneous and incompressible body $B\in\Re^3$, and W* a poly-convex stored energy density, coercive and such that $\det \boldsymbol{F} \to 0$ when $W^*\to\infty$. Let $\mathcal{A}$ the admissible deformation set and $\mathcal{E}(u)$ the total energy functional. If $\exists \inf_{u\in\mathcal{A}} \mathcal{E}(u) < +\infty \Rightarrow \exists u^* \in\mathcal{A}$ such that

$$\mathcal{E}(u^*) = \inf_{u\in\mathcal{A}} \mathcal{E}(u)$$

Physical aspects. Phase transitions models

The non convex variational problem, with two-well structure, was first studied by Van der Wall for a compressible fluid whose free energy at constant temperature, depends not only by the density, but also from density gradient.

Many authors have been developed multy-well energy model in different applications. For example recalling Ball & James [5] in austenite-mar

transitions, Bhattacharya & James[6] for shape memory alloy, De Simon Podio Guidugli [7] in ferromagnetic materials, Zanzotto [8] in crystalline materials, Conti & De Simone [9] polymers and rubber, Del Piero[10 Truskinovsky[11], Fosdick[12], Royer[13], Gurtin[14], in structural me Braides [15], Fonseca[16], Conti[17], in mathematical point of view.

When W has multi-well then the minimum problem may not hav solutions then, minimizing sequence develop finer and finer oscillations that ca be interpreted as fine scale microstructure like as observed in many materials

This force it to make very fine mixtures and the last concept is an op problem because, today, classify all possible microstructure of a materials exhaustive problem.

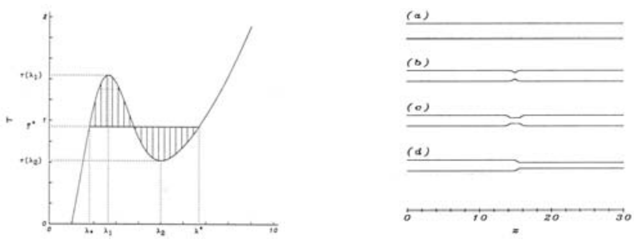
5.1. By-phase materials

If the stored energy form is typical no-convex then the stress derived has monotone envelope and this represents the real response of the mater behaviour.

In fig. 1 is possible to see deformation discontinuity in the typical polymers, necking and drawing, phenomena with two graphs in comparison.

In the first one, a non monotone law can be characterized of multiple deformations with equal stress (i.e. Maxwel stress).

In the latter graph, we see the strict concentration deformation zone where plastic part appears respect the other elastic part.



(a) (b)

Figure 1. (a) Non monotone stress-strain (b) Two-phases co-existence

In other words, inside to the material two-distinct phases are present and these represents the re-arrangement of the microstructure.

From analytical point of view equilibrium problem must be characterize energy global minimum research and, particularly, stable or metastable the stress function.

Another form, more simple respect to the relaxation, is to make a minim sequence such that energy value has minimum, when the sequence limit it tends to infinity.

6 Example solutions in 1-D case

Now, we consider an incompressible elastic bar undergoing to simple acti as well as, in classical mechanics models (i.e. d.S. Venant's model).

Generally, we consider one-dimensional body $B : \{x \mid x \in 0,1\}$ undergoing either hard or soft device with a poly-convex stored energy.

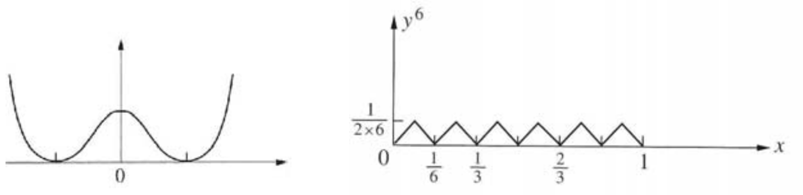
In the following, we regard axial, flexure and torsion action over B proposing a solutions by means simple sequence such that energy minimum verified.

6.1 *Axial force*

This case was first studied by Ericksen in the famous paper [19] where he solved hard and soft device conditions through energetic considerations only, but not considering minimizing sequence. Here we consider a no-convex energy as in Figure 2 whose analytical expression has the form $(u) = (u^2-1)^2$

Then, we build the sequence, according to the minimum conditions, putting the form

$$\{u_n\} = x \qquad \textbf{if} \quad 0 < x < 1/2n$$
$$\{u_n\} = 1/n - x \qquad \textbf{if} \quad 1/2n \le x \le 2/2n$$



(*a*) (*b*)

Figure 2 . (*a*) *The no-convex energy graph* (*b*) *Minimizing sequence graph*

The graph of the minimizing sequence has the form in Figure 2b.

Then, to determine the total energy we performed a simple integration

$$\mathcal{E}(u) = n \int_0^{1/2n} x^2 + \int_{1/2n}^{2/2n} (1/n - x^2)$$

Passing to the limit we find

$$lim_{\, n\to\infty} \; \mathcal{E}(u_n) = 0$$

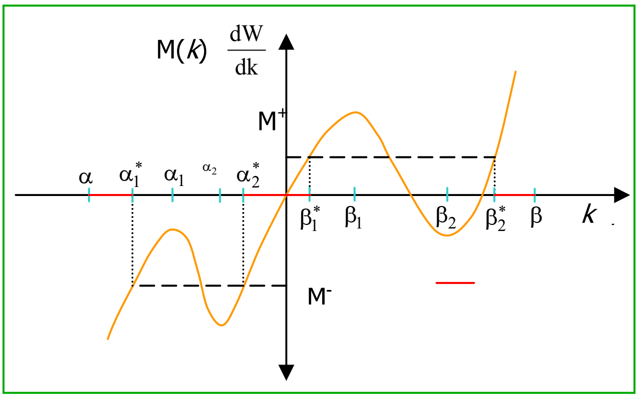
and so problem conditions and restraints are verified. Follow the pictures comparison with experimental results.[20]

6.2 *F*lexure

In this case we restrict our attention over a particular functions class whose form is no-convex type with non monotone derivate (see fig. 3). Under this assumption the stored energy has dependence by curvature parameter $k = \omega'$ and the bending moment is derived by differentiation of $W(k)$.

The property description of $W(k)$ are folowing

$$
\begin{array}{ll}
W(k) : [\alpha, \beta] \rightarrow \Re^1 & W(k) \in \mathbf{C}^2[\alpha\beta] \\
W(k)_{kk} > 0 & k \in (\alpha, \alpha_1) \cup (\alpha_2, \beta_1) \cup (\beta_2, \beta) \\
W(k)_{kk} < 0 & k \in (\alpha_1, \alpha_2) \cup (\beta_1, \beta_2) \\
W_{k|\beta} \geq W_{k|\beta 1} & W_{k|\alpha} \leq W_{k|\alpha 2} \\
W(k)_{|0} = 0 &
\end{array}
$$



*F*igure 3. Non monotone bending moment – curvature relationship

We build a minimizing deformations sequence by means of the rotation parameter ω and consider the sequence $\omega_n \in \mathbf{C}^1[0, L] \rightarrow \Re$ as

$$
\begin{array}{ll}
\omega_n(x) = 0 & \text{if } 0 \leq x \leq x^\circ \\
\omega_n(x) = \omega^\circ (x-x^\circ)/(1-n) & \text{if } x^\circ \leq x \leq (x^\circ + 1/n) \\
\omega_n(x) = \omega^\circ & \text{if} (\, x^\circ + 1/n) < x \leq L
\end{array}
$$

The sequence limit, for $n \rightarrow \infty$, tend to jump function with value equal to ω° when $x = x^\circ$.

Passing to the energy functional $\mathcal{E}(\omega')$ we have for $\omega^\circ > 0$

$$\lim_{n\to\infty} \int_L \mathcal{E}(\omega_n'(x)dx) = \lim_{n\to\infty} \mathbf{M}^\circ\, \omega^\circ/n^{-1}\,(1/n) = \mathbf{M}^\circ\omega^\circ$$

and for $\omega^\circ < 0$

$$\lim_{n\to\infty} \int_L \mathcal{E}(\omega_n'(x)dx) = \lim_{n\to\infty} -\mathbf{M}^\circ\, \omega^\circ/n^{-1}\,(1/n) = -\mathbf{M}^\circ\omega^\circ$$

In other words, it is possible to consider a discontinuity rotation field as minimum to the functional $\mathcal{E}(\omega')$**. The limit is coincident with inf** $\mathcal{E}(\omega')$**, in the admissible functions set, and so, the formulate sequence is minimizing.**

6.3 Torsion

Through their experimental results, Nadai[22], Nakanishi[23] discovery a duplex deformation concentration manner namely, longitudinal and cross with different microstructure formation.

From equilibrium point of view Fosdick & Zangh [24] have been developed the minimum problems in 3-D cylindrical solids. Successively Rizzoni's PhD-thesis [25] was investigated the equilibrium configuration as energy minimum over an elastic incompressible cylindrical model

In this case the configuration body attain minimum energy choosing among two minimizing sequence (bi-phase solution) and the universal solution (one-phase solution).

Here we consider a mono-dimensional homogeneous elastic bar inside the reference system. Let us Ω_B $[0, r)\times[0,2\pi)\times(0, Z)$ **the configuration reference and let us u**$[\mathcal{F}(r^f, \varphi^f, Z^f), \mathcal{G}(r^f, \varphi^f, Z^f), \mathcal{K}(r^f, \varphi^f, Z^f)]$ **the displacement field over the body** B. **The isochoric deformation field f assume the form for components**

$$[(r^f+\mathcal{F}(r^f, \varphi^f, Z^f)\,,\ \varphi^f+\mathcal{G}(r^f, \varphi^f, Z^f),\ L^f+\mathcal{K}(r^f, \varphi^f, Z^f)\,]$$

and again we assume $\mathcal{F} = \mathcal{F}(r)$**.**

The imposed boundary conditions affirm that $\forall r \in (0, r_{max})$ **and** $\forall\varphi \in (0, 2\pi)$

$$\int_L (r + \mathcal{F}(r))\, \mathcal{G}_Z\, (r^f, \varphi^f, Z^f)dZ = \omega Lr$$

Under the hypothesis of null body force and \omega the torsion angle, then the equilibrium conditions are mono-phase minimum solutions for the total energy functional.

$$\mathcal{E} = 2\pi L \int_r w(\omega\, r) r\, dr$$

In the by-phase configuration, for assigned torsion angle, three concentric zone are detected Ω_1 **cylindrical set,** Ω_{12} **ring set,** Ω_2 **external ring set.**

Let us a deformation measure as $k(f) = (\nabla f \nabla f - 3)^{1/2}$ **then in** Ω_1 **when** $r_1 < k_1/\omega$ **the deformation coincide with universal solution. In** Ω_2 **if** $\omega > k/ r_{max}$ **again the universal solution appear. If** $\omega < k_1/r_{max}$ **resulting only mono phase solution.**

After opportune geometric partition (n = **number of elementary cylinder in the global cylinder**, L = **cylinder length and** m = 1….n), **passing to the build the minimizing sequence we have the follow assigned deformations.**

-cross mode

$\varphi = \varphi^i + \omega \mathbf{Z} - \chi(r)(\mathbf{Z} - (m-1)L/n)$

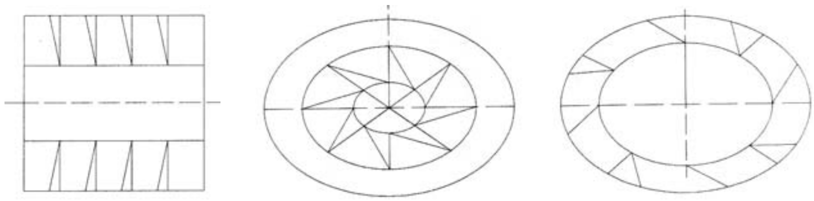
$\varphi = \varphi^i + \omega \mathbf{Z} - \gamma(r)(mL/n - Z)$

-longitudinal mode

$\varphi = \varphi^i + \omega \mathbf{Z} - \xi(\mathbf{r})(\varphi - 2(m-1)\pi/\mathbf{n})$ $\mathbf{Z} = \mathbf{Z} - \omega^{-1}\xi(r)(\varphi^i - 2(m-1)\pi/n)$

$\varphi = \varphi^i + \omega \mathbf{Z} - \zeta(\mathbf{r})(2m\pi/n - \varphi^i)$ $\mathbf{Z} = \mathbf{Z} - \omega^{-1}\zeta(r)(2(m\pi/n - \varphi^i)$

In the folow fig. 4 the minimizing sequence graph appear and in the reference[8] **we see the comparison with experimental results.**



(*a*) (*b*)

Figure 4. Minimizing sequence graph: (a) cross mode (b) longitudinal mode

Concusions and future issue

The tool derived by analytical solution of the minimum problem, where stored energy is no-convex, represents a good framework to model the behaviour of materials; these numerical results match the experimental ones.

In fact, by means of by-phase concept, or deformation concentration, we can model inelastic problems in the finite elasticity context.

Up-grade of our model must be considered in order to introduce interfacial energy contribution and hence enriching total energy functional with more terms.

Further steps must involve fracture and two-dimensional models.

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RENDICONTI DEL CIRCOLO MATEMATICO DI PALERMO
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On Finite Sphere Coverings in 3-Space

Paolo Canova Pier Mario Gandini Paola Gramaglia

Abstract

We show that in euclidean 3-space for each integer $k \geq 744$ there is a nonlinear arrangement of k unit balls which covers a convex body of volume greater than the volume of the convex body covered by any linear arrangement with the same number of balls.

MSC 2000: 52C17, 05B40

1 Introduction

A basic problem in the theory of finite sphere coverings in euclidean d-space E^d is to determine, for a positive integer k, the maximal volume of all convex bodies which can be covered by k unit balls.An important role in the study of this problem is played by linear arrangements of unit balls i.e. by arrangements of unit balls whose centers lie on a straight line. It was shown that among all linear arrangements of balls a convex body of maximal volume is covered if and only if the distance between two consecutive centers is $\frac{2}{\sqrt{d}}$. We shall refer to this as to the best linear arrangement.

It was conjectured that for $d \geq 5$ the best linear arrangement is best possible also among all arrangements [2]. Although this conjecture is supported by some partial results, the problem is still open.

In dimension 3 and 4 the situation is different; it seems that there are two integers K_0^d and K_1^d (d=3,4 $K_0^d \leq K_1^d$) such that:

i) for $k < K_0^d$ linear arrangements are best possible;

ii) for $k = K_0^d$ and for $k \geq K_1^d$ there is a nonlinear arrangement of k unit balls that covers a convex body of volume greater than the volume of the convex body covered by the best linear arrangement with k balls.

iii) for some values of k (but not for all) such that $K_0^d < k < K_1^d$ linear arrangements are best possible.

It has been proved that $K_0^d \leq 548$ [1].

In this paper we show that $K_1^d \leq 744$.

2 The construction

To prove the result we use the construction already done in [1]. Let C be the cube of edge-length $\frac{4}{\sqrt{5}}$ which can be covered by nine unit balls. Then for n, m, p positive integers, we consider the box $C_{n,m,p}$ consisting of nmp copies of C whose volume is $nmp\Big(\frac{4}{\sqrt{5}}\Big)^3$ and which is covered by $nmp + (n+1)(m+1)(p+1)$ unit balls.

Let now h be a nonnegative integer such that $h+1 \leq \frac{n}{2}, \frac{m}{2}, \frac{p}{2}$. Then on each of the three edges relative to a vertex A_i (i= 1,...,8) of $C_{n,m,p}$ we choose the point P_{ij} such that $d(A_i, P_{ij}) = \frac{4h+3}{\sqrt{5}}$. We assume that P_{ij} belongs to the edge of vertices A_i and A_j. Let now A_iA_j, A_iA_r,A_iA_s, A_jA_t, A_jA_u the edges relative to the vertices A_i and A_j. We cut $C_{n,m,p}$ by means of the plane parallel to A_iA_j and containing the points P_{ir}, P_{is},P_{jt},P_{ju}. If we do this for each edge of $C_{n,m,p}$ we obtain a truncated parallelepiped denoted by $C^h_{n,m,p}$.

If the truncation is done for the four edges of length p and for k edges ($k = 1, 2, 3, 4$) of length m we denote this truncated parallelepiped by $C^{\emptyset,k\times h,4\times h}_{n,m,p}$.

Notations. If P is one of the previous $C^h_{n,m,p}$,$C^{\emptyset,k\times h,4\times h}_{n,m,p}$,then $Vol(P)$ is its volume, $G(P)$ is the number of balls of the covering of the 3-space generated by the nine balls which cover C, which have center in P and cover it. $S(G(P))$ is the best linear arrangement with $G(P)$ unit balls and $Vol(S(G(P)))$ is its volume.

We remark that the best linear arrangement with k units balls covers a convex body (a sausage) of volume $2,4184k + 0.96426$.

It has been shown [1] that :

$$VolC^h_{n,m,p} = \Big(\frac{4}{\sqrt{5}}\Big)^3 nmp - 4(n+m+p)\frac{2(4h+3)^2}{5\sqrt{5}} + \frac{24(4h+3)^3}{15\sqrt{5}} - \frac{8(4h+3)^3}{20\sqrt{5}}$$

$$\begin{aligned} G(C^h_{n,m,p}) &= nmp + (n+1)(m+1)(p+1) \\ &\quad -2(h+1)(h+2)(n+m+p-6h-3) \\ &\quad -2h(h+1)(n+m+p-6h) \\ &\quad -8[h^3 + (h+1)^3 - \frac{h(h+1)(2h+1)}{6} \\ &\quad -\frac{h(h-1)(2h-1])}{3} + \frac{h^2(h-1)}{2}] \end{aligned}$$

In an analogous way it is also possible to show that:

$$\begin{aligned} VolC^{\emptyset,k\times h,4\times h}_{n,m,p} &= \Big(\frac{4}{\sqrt{5}}\Big)^3 nmp - 4p\frac{2(4h+3)^2}{5\sqrt{5}} - km\frac{2(4h+3)^2}{5\sqrt{5}} \\ &\quad +2k\frac{(4h+3)^3}{15\sqrt{5}}.(k=1,2,3,4). \end{aligned}$$

$$\begin{aligned} G(C^{\emptyset,k\times h,4\times h}_{n,m,p}) &= nmp + (n+1)(m+1)(p+1) \\ &\quad -4(p-2h-1)\frac{(h+1)(h+2)}{2} \\ &\quad -k(m-2h-1)\frac{(h+1)(h+2)}{2} - 4h(h+1)\frac{p-2h}{2} \\ &\quad -kh(h+1)\frac{m-2h}{2} - 2k[h^3+(h+1)^3 - \sum_1^h i^2 - \sum_1^{h-1} i^2] \\ &\quad -2(4-k)[h^3+(h+1)^3-(h+1)^2\frac{h}{2}-(h-1)\frac{h^2}{2}] \\ &\quad (k=1,2,3,4) \end{aligned}$$

3 The main result

Theorem.- *For each $k \geq 744$ there is a non linear arrangement of k unit balls which covers a convex body of volume greater then the volume of the sausage covered by the best linear arrangement with k balls.*

Proof.-A first bound for K^3_1 can be found by considering cubes C_n of edges $\frac{4n}{\sqrt{5}}$ and looking for the values of n such that:

$$Vol(S(G(C_n))) < Vol(C_{n-1}) \tag{1}$$

In this way the theorem will be proved for each k such that:

$$G(C_{n-1}) \leq k < G(C_n).$$

Since C_n is covered by $n^3+(n+1)^3$ unit balls (1) is equivalent to

$$2,4184[n^3+(n+1)^3]+0,96426 < \Big(\frac{4}{\sqrt{5}}\Big)^3(n-1)^3$$

i.e.

$$f(n) = 0,8875n^3 - 24,4282n^2 + 9,9178n - 9,10699 > 0.$$

The function $f(n)$ has a maximum in $n = 0,2053$, a minimum in $n = 18,1445$, is increasing for $n > 18,1445$ and positive for $n \geq 28$. Therefore, for each $k \geq 46341 = G(C_{28})$ the best linear arrangement is not the best possible.

Now we improve this first bound of K_1^3 by considering a sequence

$$P_1, P_2, ..., P_m$$

of polytopes such that $P_1 = C_{28}$ and for $i = 1, ..., m$

$$\begin{aligned} &G(P_i) > G(P_{i+1}) \\ &Vol(S(G(P_i))) < Vol(P_{i+1}). \end{aligned}$$

So the Theorem will be proved for each k such that $G(P_{i+1}) \leq k < G(P_i)$ and then for $k \geq G(P_m)$.

In the first part of the sequence the polytopes are suitable $C_{n,m,p}$, in the following the polytopes are also $C^h_{n,m,p}$and $C^{\emptyset,k\times h,4\times h}_{n,m,p}$. For these ones we use the formulae found in the construction.

In the following table in the first column there is the polytope P_i, in the second the volume of P_i which for $i > 1$ must be greater then the value of $S(G(P_{i-1}) - 1)$, in the third $G(P_i)$,which is the new bound for K_1^3, in the last the volume of $S(G(P_i))$. From the last row of the table the Theorem follows.

P_i	$Vol(P_i)$	$G(P_i)$	$Vol(S(G(P_i))$
C_{28}	$125660,49$	46341	$112072,04$
$C_{28,25,28}$	$112196,86$	41466	$100282,34$
$C_{25,26,27}$	$100461,99$	37206	$89979,95$
$C_{21,25,30}$	$90158,19$	33482	$80973,83$
$C_{22,23,28}$	$81102,30$	30176	$72978,60$
$C_{20,22,29}$	$73042,45$	27250	$65902,36$
$C_{20,24,24}$	$65944,28$	24645	$59602,43$
$C_{16,21,31}$	$59624,62$	22384	$54134,43$
$C_{14,26,26}$	$54175,05$	20399	$49333,90$
$C_{15,23,25}$	$49372,34$	18609	$45004,97$
$C_{18,19,23}$	$45027,57$	16986	$41079,91$
$C_{18,19,21}$	$41112,13$	15542	$37587,74$
$C_{13,22,23}$	$37654,64$	14306	$34598,59$
$C_{16,18,21}$	$34620,74$	13154	$31812,60$
$C_{14,19,21}$	$31976,10$	12186	$29471,59$
$C_{14,16,23}$	$29491,74$	11272	$27261,17$

$C_{14,18,19}$	$27408,09$	10488	$25365,14$
$C_{13,18,19}$	$25450,37$	9766	$23619,06$
$C_{12,15,23}$	$23698,72$	9132	$22085,79$
$C_{12,14,23}$	$22118,81$	8544	$20663,77$
$C_{10,19,19}$	$20664,83$	8010	$19372,35$
$C_{11,14,22}$	$19394,03$	7528	$18206,68$
$C_{12,14,19}$	$18272,06$	7092	$17152,26$
$C_{10,15,20}$	$17172,99$	6696	$16194,57$
$C_{9,15,21}$	$16288,47$	6355	$15369,90$
$C_{12,14,16}$	$15386,99$	6003	$14518,62$
$C_{13,14,14}$	$14585,59$	5698	$13781,01$
$C_{11,11,20}$	$13852,97$	5444	$13166,73$
$C_{12,12,16}$	$13188,85$	5177	$12521,02$
C_{13}	$12576,35$	4941	$11950,27$
$C_{10,11,19}$	$11963,84$	4730	$11439,99$
$C_{11,13,14}$	$11460,10$	4522	$10936,97$
$C_{10,12,16}$	$10990,71$	4351	$10523,42$
$C_{11,12,14}$	$10578,56$	4188	$10129,22$
$C_{9,11,18}$	$10200,75$	4062	$9824,50$
C_{12}	$9891,64$	3925	$9493,18$
$C_{8,13,16}$	$9525,28$	3806	$9205,39$
$C_{9,12,15}$	$9273,41$	3700	$8949,04$
$C_{11,11,13}$	$9004,37$	3589	$8680,60$
$C_{9,13,13}$	$8706,70$	3481	$8419,41$
$C_{9,11,15}$	$8500,63$	3405	$8235,62$
$C_{10,12,12}$	$8243,03$	3299	$7979,26$
$C_{10,10,14}$	$8014,06$	3215	$7776,12$
$C_{8,10,17}$	$7785,08$	3142	$7599,58$
C_{11}	$7619,08$	3059	$7398,85$
$C_{9,12,12}$	$7418,73$	2986	$7222,31$
$C_{7,13,14}$	$7292,79$	2954	$7144,92$
$C_{9,10,14}$	$7212,65$	2910	$7038,51$
$C_{8,11,14}$	$7052,37$	2852	$6898,24$
$C_{10,11,11}$	$6926,43$	2794	$6757,97$
$C_{9,11,12}$	$6800,50$	2748	$6646,73$
$C_{9,10,13}$	$6697,46$	2710	$6554,83$
$C_{8,12,12}$	$6594,42$	2673	$6465,35$
$C_{9,9,14}$	$6491,39$	2634	$6371,03$

$C_{8,10,14}$	$6411,24$	2605	$6300,89$
$C^5_{14,14,14}$	$6339,16$	2435	$5889,77$
$C^3_{12,12,12}$	5906.98	2285	$5527,01$
$C^5_{14,14,13}$	$5595,71$	2158	$5219,87$
$C^3_{12,12,11}$	$5243,67$	2036	$4924,83$
$C^5_{14,13,13}$	$4932,41$	1910	$4620,11$
$C^3_{12,11,11}$	$4649,05$	1812	$4383,11$
$C^2_{14,9,9}$	$4435,11$	1746	$4223,49$
$C^4_{12,12,12}$	$4273,39$	1665	$4027,60$
$C^3_{12,11,10}$	$4054,44$	1588	$3841,38$
$C^3_{13,11,9}$	$3865,53$	1519	$3674,51$
$C^4_{12,12,11}$	$3707,40$	1452	$3512,48$
$C^3_{12,10,10}$	$3528,51$	1389	$3360,12$
$C^1_{13,8,7}$	$3369,66$	1356	$3280,31$
$C^2_{11,9,9}$	$3303,83$	1311	$3171,49$
$C^4_{12,11,11}$	$3210,10$	1264	$3057,82$
$C^3_{11,10,10}$	$3117,08$	1232	$2980,43$
$C^3_{12,10,9}$	$3002,60$	1190	$2878,86$
$C^2_{13,9,7}$	$2891,68$	1159	$2803,89$
$C^2_{11,9,8}$	$2823,71$	1128	$2728,92$
$C^4_{11,11,11}$	$2775,76$	1099	$2658,79$
$C^3_{10,10,10}$	$2705,64$	1075	$2600,74$
$C^2_{12,9,7}$	$2617,63$	1052	$2545,12$
$C^3_{12,9,9}$	$2545,36$	1016	$2458,06$
$C^2_{10,9,8}$	$2498,13$	1002	$2424,20$
$C^2_{13,8,7}$	$2457,35$	992	$2400,02$
$C^2_{11,8,8}$	$2406,55$	968	$2341,97$
$C^2_{11,9,7}$	$2343,58$	945	$2286,35$
$C^3_{10,10,9}$	$2294,20$	918	$2221,06$
$C^3_{11,9,9}$	$2242,69$	899	$2175,11$
$C^3_{11,10,8}$	$2179,72$	876	$2119,48$
$C^2_{10,8,8}$	$2126,77$	859	$2078,37$
$C^3_{12,9,8}$	$2088,13$	842	$2037,26$
$C^2_{10,9,7}$	$2069,53$	838	$2027,58$
$C^{\emptyset,4\times3,4\times3}_{9,8,10}$	$2028,56$	830	$2008,24$
$C^2_{7,7,13}$	$2023,02$	825	$1996,14$
$C^{\emptyset,4\times2,4\times2}_{7,9,9}$	$2004,71$	823	$1991,31$

$C^2_{7,8,11}$	$1989,39$	808	$1955,03$
$C^4_{10,10,11}$	$1970,07$	792	$1916,34$
$C^3_{9,9,10}$	$1940,01$	782	$1892,15$
$C^3_{8,8,13}$	$1904,95$	774	$1872,81$
$C^3_{8,10,10}$	$1882,77$	761	$1841,37$
$C^2_{8,8,9}$	$1846,99$	750	$1814,76$
$C^3_{8,9,11}$	$1836,97$	744	$1800,25$

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RENDICONTI DEL CIRCOLO MATEMATICO DI PALERMO
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APPROCHS TO SECURITY PROBLEMS BY UTILISING ALGEBRAIC TOOLS

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Abstract. We give the construction of the jacobian dual of a module and we present possible applications of this algebraic object.

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Introduction

A little comment to explain the meaning of the word security in the title is necessary. An ongoing project of NATO entitled "SECURITY THROUGH SCIENCE" starting from 2000, concerns the security related to science domain. This program deals extensively with different themes as terrorism and spy; in substance ,with the security of a region or, in general, of the world. Could the mathematical sciences help in this problem?
An interesting suggestion is given by the following situation: a graph G gives the connection between places of a country that are seats of chemical arms. A possible mathematical approach could be: one chooses a graph G such that the presentation matrix of its edge ideal $I(G)$ is a linear matrix (of variables). Then one gives, instead of the graph G, that is the edge ideal $I(G)$, a different mathematical object linked to G, for example the jacobian dual module $I(G)^\wedge$ (not an ideal in general). If $I(G)^\wedge$ is yet an ideal, we would obtain a new graph $G^\wedge$, so called jacobian dual graph of G.
Starting from the $I(G)^\wedge = I(G^\wedge)$, the mathematician reconstructs the graph.
We remark that the procedure is not simple. It involves at last two steps: 1.The presentation matrix of $I(G)$; 2.The symmetric algebra of $I(G)$. There are graphs whose the presentation matrix of edges ideal is not linear, but many classes are good for this problem.
In this paper we present the construction of the dual module $M^\wedge$ of a finitely generated module M on a polynomial ring in a finite number of variables over a field k . Since we are interested to properties that remain, we consider particular linear matrices that produce circuits. In this direction, in the N.1 we give a new definition for circuits in the general sense, inspired by [2].

In N.2, we define the jacobian dual and we give examples and applications, following the main aim of NATO.

1. Circuits

The notion of circuit is very important in linear algebra, the object of engineering sciences.
It comes from matroid theory, but it is very important in the research of an universal Gröbner basis for the ideal I, the vanishing ideal of $k[X_1, \dots, X_n]$ of a $(n-m)$- dimensional vector subspace of k^n.
We give some definitions:
Let V be an $(n-m)$- dimensional vector subspace of k^n and let I be its vanishing ideal in the ring $k[X_1, \dots, X_n]$, generated by m linear forms $f_1, \dots, f_m$, $f_i = \sum_{j=1}^{n} a_{ij}X_j$, $i = 1, \dots, m$.

Definition 1.1. *Let $f \in I$ be a non-zero linear form.$f = \sum_{i=1}^{n} a_i X_i$ is a circuit if supp$(f) = \{i : a_i \neq 0\}$ is minimal with respect to inclusion.*

There are at most $\binom{n}{m-1}$ circuits, since if I is generated by linear forms, the Buchberger algorithm is equivalent to Gaussian elimination on the coefficients matrix.

Proposition 1.2. *Let I be the vanishing ideal in $k[X_1, \dots, X_n]$ of a $(n-m)$-dimensional vector space. Then every reduced Groöbner basis consists of m circuits.*

Theorem 1.3. ([5], *chap.*2)
Let I be an ideal of $k[X_1, \dots, X_n]$ generated by linear forms. The set of circuits in I is a minimal universal Gröbner basis for I.

Now consider any commutative, noetherian, ring R with unit element.
Consider the ring $S = R[Y_1, \dots, Y_n]$, where $Y_1, \dots, Y_n$ are variables on R and m elements $f_1, \dots, f_m \in S$ that are linear in the $Y_i's$ variables.
$f_i = \sum_{j=1}^{n} a_{ij}Y_j$, $i = 1, \dots, m$, $a_{ij} \in R$.
Let $J = (f_1, \dots, f_m)$ be the vanishing ideal of these forms in S. The ideal J is known and it appears as the presentation ideal of the symmetric algebra $Sym_R(M)$ of a module M that is the cokernel of the map $0 \to R^m \xrightarrow{f} R^n$, where f is represented by the matrix (a_{ij}). (In fact, we can obtain the ideal J, as a presentation ideal of the symmetric algebra of a vector space.)
However the linear forms have coefficients in the ring R, not in the field k(linear algebra). The aim of our research is to have:

1. A definition of circuit in this context.
2. If $R = k[X_1, \dots, X_s]$, to prove that if J is an ideal generated by linear forms of $R[Y_1, \dots, Y_n]$ that are circuits, then this set must be a minimal universal Gröbner basis of J.

Consider the presentation of $Sym_R(M)$

$$Sym_R(M) = R[Y_1, \dots, Y_n]/J.$$

Let $<$ be a monomial order on the monomials of $R[Y_1, \dots, Y_n]$ in the variables Y_i such that

$$Y_1 < Y_2 < \dots < Y_n.$$

We call $<$ an admissible order.
With respect to this term order, if $f = \sum a_\alpha \underline{Y}^\alpha$, $\underline{Y}^\alpha = Y_1^{\alpha_1} \cdots Y_n^{\alpha_n}$, $\underline{\alpha} \in \mathbb{N}^n$, we put $\text{in}_< f = a_\alpha \underline{Y}^\alpha$, where $\underline{Y}^\alpha$ is the largest monomial in f such that $a_\alpha \neq 0$.
If we assign degree 1 to each variable Y_i and degree 0 to the elements of R, we have the following facts:

1) J is a graded ideal
2) The natural epimorphism $S \to Sym_R(M)$ is a graded homomorphism of graded algebras on R, S is a graded ring and $Sym_R(M)$ is a graded algebra.

Definition 1.4. *The ideal $(f_1, \dots, f_n)$ is generated by circuits if*

$$in_< J = (I_1Y_1, I_2Y_2, \dots, I_nY_n),$$

where $I_1, \dots I_n$ are ideals generated by elements of R.

If $R = k[X_1, \dots, X_s]$ we can use the Gröbner bases theory and Buchberger's algorithm to compute $\text{in}_< J$.
$Sym_R(M) = k[X_1, \dots, X_s, Y_1, \dots, Y_n]/J$. We can introduce a term order on $S = k[X_1, \dots, X_s, Y_1, \dots, Y_n]$, such that $Y_1 < Y_2 < \dots < Y_n$ and $X_i < Y_i$ for any i.
For example $X_1 < X_2 < \dots < X_s < Y_1 < Y_2 < \dots < Y_n$ is such a term order.
If G is a Gröbner basis for $J \subset k[X_1, \dots, X_s, Y_1, \dots, Y_n]$, we have $\text{in}_< J = (\text{in}_< G) = (\text{in}_< f, f \in J)$ and if the elements of G are linear in the $Y_i s$, it follows that $f_1, \dots, f_n$ is generated by circuits.

Example 1.5. *For $s = n$, let J be the ideal of S generated by the 2-minors of the matrix*

$$\begin{pmatrix} X_1 & X_2 & \cdots & X_n \\ Y_1 & Y_2 & \cdots & Y_n \end{pmatrix}$$

Then $in_< J = (I_1Y_1, \dots, I_nY_n) = ((X_1)Y_2, (X_1, X_2)Y_3, \dots, (X_1, X_2, \dots, X_{n-1})Y_n)$ and J is generated by circuits. The set $G = \{X_1Y_2 - X_2Y_1, X_1Y_3 - X_3Y_1, \dots, X_{n-1}Y_n - X_nY_{n-1}\}$ is a minimal universal Gröbner basis for J.

Remark 1.6. *If $R = k[X_1, \dots, X_s]$, from the theory of Gröbner basis, if $f_1, \dots, f_n$ is generated by circuits with respect to any admissible term order $<$, then $f_1, \dots, f_n$ is generated by circuits for another admissible term order, too.*

2. Jacobian dual

Let $R = k[X_1, \ldots, X_s]$ be a polynomial ring and let E be a finitely generated R-module with presentation:

$$R^m \xrightarrow{\phi} R^n \to E \to 0$$

where the entries of the $n \times m$ matrix $A = (a_{ij})$ that represents ϕ are homogeneous linear forms.
The equations of the symmetric algebra of E, $\mathrm{Sym}_R(E) = S(E)$ are

$$f_j = \sum_{i=1}^{n} a_{ij} Y_i \qquad j = 1, \ldots, m.$$

There is a naive duality for $S(E)$, if we write the equations f_j in the X_i's variables

$$f_j = \sum_{i=1}^{n} a_{ij} Y_i = \sum_{i=1}^{s} b_{ij} X_i \qquad j = 1, \ldots, m,$$

where $B = (b_{ij})$ is an $s \times m$ matrix of homogeneous linear forms in the Y_i's variables.

We have:

$$A^t \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix} = B^t \begin{pmatrix} X_1 \\ \vdots \\ X_s \end{pmatrix} = \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}.$$

Now we put $Q = k[Y_1, \ldots, Y_n]$ and consider the cokernel N of the map

$$Q^m \xrightarrow{\Psi} Q^s \to N \to 0,$$

where Ψ is the map represented by B and N defines the Jacobian dual module of E ([3], [4]).

Example 2.1. *We can write the relation* $f = (X_1 - 2X_2)Y_1 + (X_1 + X_2)Y_2 + X_3Y_3$ *as* $f = (Y_1 + Y_2)X_1 + (-2Y_1 + Y_2)X_2 + Y_3X_3$.

Remark 2.2. $\mathrm{Sym}_R(E) \cong \mathrm{Sym}_Q(N)$.

Example 2.3. *Suppose that* $A \cong B$*, in the sense that the two matrices* A *and* B *have the same elements under the substitution* $X_i \to Y_i$, $n = s$. *Then* $R \cong Q$ *and* $E \cong N$.

There is a nice situation that will be interesting in the following.
Let $R = k[X_1, \ldots, X_n]$, $I = m_+ = (X_1, \ldots, X_n)$, $\mathrm{Sym}_R(m_+) = \mathcal{R}(m_+) = K[X_1, \ldots, X_n; Y_1, \ldots, Y_n]/J$, where J is generated by the binomials $X_iY_j - X_jY_i$, $1 \le i < j \le n$, the 2×2-minors of the $2 \times n$ matrix

$$\begin{pmatrix} X_1 & X_2 & \ldots & X_n \\ Y_1 & Y_2 & \ldots & Y_n \end{pmatrix}.$$

The binomials in the X_i's give the dual matrix B of the relation matrix A of m_+ under the substitution $X_i \to Y_j$, $i, j = 1, \ldots, n$.
Notice that the set of binomials is an universal Gröbner basis for the ideal J

Example 2.4. *Consider the matrix B of indeterminates*

$$\begin{pmatrix} X_{11} & X_{12} & 0 & 0 \\ 0 & X_{22} & X_{23} & 0 \\ 0 & 0 & X_{33} & X_{34} \end{pmatrix}$$

Put $R = k[X_{11}, X_{12}, X_{22}, X_{23}, X_{33}, X_{34}]$ and consider the cokernel M of the map

$$R^4 \xrightarrow{B} R^3 \to M \to 0$$

$$Sym_R(M) = R[Y_1, Y_2, Y_3]/(f_1, f_2, f_3)$$

$f_1 = X_{11}Y_1 + X_{12}Y_2$
$f_2 = X_{22}Y_2 + X_{23}Y_3$
$f_3 = X_{11}Y_3 + X_{34}Y_4$
$J = (f_1, f_2, f_3)$.

The dual matrix $B^\wedge$ of B is

$$\begin{pmatrix} Y_1 & 0 & 0 \\ Y_2 & Y_2 & 0 \\ 0 & Y_3 & Y_3 \\ 0 & 0 & Y_4 \end{pmatrix}$$

and the jacobian dual N is

$$Q^3 \xrightarrow{\hat{B}} Q^4 \to N \to 0$$

$$Q = C[Y_1, Y_2, Y_3, Y_4]$$

Question:
Are f_1, f_2, f_3 circuits in the generalized sense? We have to compute:
1) *$in(J)$ with to respect the order of variables $Y_1 > Y_2 > Y_3 > X_{11} > \; > X_{34}$*

2) *in the same way $in(J)$ with to respect the order of the variables $X_{11} > X_{12} > \ldots > X_{34} > Y_1 > Y_2 > Y_3$.*
For the first order in $f_1 = X_{11}Y_1, inf_2 = X_{22}Y_2, inf_3 = X_{33}Y_3$ and J admits a Gröbner basis linear in the Y_i variables. For the second order, we have $inf_1 = X_{11}Y_1, inf_2 = X_{22}Y_2, inf_3 = X_{33}Y_3$ and J admits again a

Gröbner basis linear in the X_{ij} variables .Then M and N have J generated by circuits.

Example 2.5. *(complicated problem)*
To choose a graph G whose the jacobian dual module $(I(G))^\wedge = I(G^\wedge)$ (in general $(I(G))^\wedge$) is not an ideal of Q), where $G^\wedge$ is the dual graph in this sense. We need $(I(G))^\wedge$ generated by monomials of degree two, to have a new graph $G^\wedge$.
Then a plane for security could be: to do the graph G but, really, one operates with the graph $G^\wedge$. For the triangle, the dual graph constructed via the jacobian dual is a new triangle obtained by a permutation of its vertices.

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TRIANGLES WITH COORDINATES OF VERTICES FROM PELL AND PELL-LUCAS NUMBERS

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ABSTRACT. In this paper we consider triangles in the plane with coordinates of points from the Pell and the Pell-Lucas sequences. For four infinite sequences of such triangles we explore how are they related to each other and what geometric properties they share.

The Pell and Pell-Lucas sequences P_n and Q_n are defined by the recurrence relations

$$P_0 = 0, \qquad P_1 = 2, \qquad P_n = 2\,P_{n-1} + P_{n-2} \ \text{ for } n \geqslant 2,$$

and

$$Q_0 = 2, \qquad Q_1 = 2, \qquad Q_n = 2\,Q_{n-1} + Q_{n-2} \ \text{ for } n \geqslant 2.$$

The numbers Q_k make the integer sequence $A002203$ from [6] while the numbers $\frac{1}{2}\,P_k$ make $A000129$.

Let k be a positive integer. Let Δ_k and Γ_k denote the triangles with vertices $A_k = (P_k,\, P_{k+1})$, $B_k = (P_{k+1},\, P_{k+2})$, $C_k = (P_{k+2},\, P_{k+3})$ and $X_k = (Q_k,\, Q_{k+1})$, $Y_k = (Q_{k+1},\, Q_{k+2})$, $Z_k = (Q_{k+2},\, Q_{k+3})$, respectively.

Our goal in this paper is to explore some common properties of the triangles Δ_k and Γ_k. Analogous infinite series of triangles with coordinates from the Fibonacci and Lucas integer sequences was studied by the first author in [3]. There is a great similarity between these two papers in statements of results and in methods of proofs.

We begin with the following theorem which shows that these triangles share the property of orthology.

Recall that the triangles ABC and XYZ are *orthologic* when the perpendiculars at vertices of ABC onto the corresponding sides of XYZ are concurrent. The point of concurrence is $[ABC,\, XYZ]$. It is well-known that the relation of orthology for triangles is reflexive and symmetric. Hence, the perpendiculars at vertices of XYZ onto the corresponding sides of ABC are concurrent at the point $[XYZ,\, ABC]$.

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By replacing in the above definition perpendiculars with parallels we get the analogous notion of *paralogic* triangles and of two points $\langle ABC,\, XYZ\rangle$ and $\langle XYZ,\, ABC\rangle$.

The triangle ABC is paralogic to its first Brocard triangle $A_b B_b C_b$ which has the orthogonal projections of the symmedian point K onto the perpendicular bisectors of sides as vertices (see [4] and [5]).

Theorem 1. *For all positive integers m and n, the following are pairs of orthologic triangles: $(\Delta_m,\, \Delta_n)$, $(\Delta_m,\, \Gamma_n)$, and $(\Gamma_m,\, \Gamma_n)$.*

Proof. It is well-known (see [1]) that the triangles ABC and XYZ with coordinates of points $(a_1,\, a_2)$, $(b_1,\, b_2)$, $(c_1,\, c_2)$, $(x_1,\, x_2)$, $(y_1,\, y_2)$, and $(z_1,\, z_2)$ are orthologic if and only if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ x_1 & y_1 & z_1 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} a_2 & b_2 & c_2 \\ x_2 & y_2 & z_2 \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

Let $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$. Note that $\alpha + \beta = 2$ and $\alpha\beta = -1$ so that the numbers α and β are solutions of the equation $x^2 - 2\,x - 1 = 0$. Since $P_j = \frac{2(\alpha^j - \beta^j)}{\alpha - \beta}$ and $Q_j = \alpha^j + \beta^j$ for every $j \geqslant 0$, when we substitute the coordinates of the vertices of Δ_m and Δ_n into the left hand side of the above criterion we get $\frac{4\,(\alpha-1)(\beta-1)(\alpha\,\beta+1)(\alpha^n\,\beta^m - \alpha^m\,\beta^n)}{\alpha-\beta}$. For the pairs $\Delta_m,\, \Gamma_n$ and $\Gamma_m,\, \Gamma_n$ we get $2\,(\alpha-1)(\beta-1)(\alpha\,\beta+1)(\alpha^m\,\beta^n + \alpha^n\,\beta^m)$ and $(\alpha-\beta)(\alpha-1)(\beta-1)(\alpha\,\beta+1)(\alpha^m\,\beta^n - \alpha^n\,\beta^m)$. From this the conclusion of the theorem is obvious because $\alpha\,\beta + 1 = 0$. $\square$

Theorem 2. *For all positive integers m the orthocenters $H(\Delta_m)$ and $H(\Gamma_m)$ of the triangles Δ_m and Γ_m and the orthology centers $[\Delta_m,\, \Gamma_m]$ and $[\Gamma_m,\, \Delta_m]$ satisfy*

$$\frac{|H(\Delta_m)[\Delta_m,\, \Gamma_m]|}{|H(\Gamma_m)[\Gamma_m,\, \Delta_m]|} = \frac{\sqrt{2}}{2}.$$

Proof. Let us use θ_b^a as a short notation for the expression $a + b\,\sqrt{2}$. Let $A = \alpha^m$ and $B = \beta^m$. Using the Binet formula for Pell and Pell-Lucas numbers it is easy to check that $H(\Delta_m)$ has the coordinates $\frac{\theta_{12}^{17}\,A^3 + \theta_1^1\,A^2\,B + \theta_{-1}^1\,A\,B^2 + \theta_{-12}^{17}\,B^3}{2\,A\,B}$ and $\frac{-\theta_5^7\,A^3 + \theta_2^3\,A^2\,B + \theta_{-2}^3\,A\,B^2 - \theta_{-5}^7\,B^3}{2\,A\,B}$. Similarly, the orthocenter $H(\Gamma_m)$ has coordinates $\frac{-\theta_{17}^{24}\,A^3 + \theta_1^2\,A^2\,B + \theta_{-1}^2\,A\,B^2 - \theta_{-17}^{24}\,B^3}{2\,A\,B}$ and $\frac{\theta_7^{10}\,A^3 + \theta_3^4\,A^2\,B + \theta_{-3}^4\,A\,B^2 + \theta_{-7}^{10}\,B^3}{2\,A\,B}$. The same method for $[\Delta_m,\, \Gamma_m]$ gives $\frac{-\theta_{12}^{17}\,A^3 + \theta_1^1\,A^2\,B + \theta_{-1}^1\,A\,B^2 - \theta_{-12}^{17}\,B^3}{2\,A\,B}$ and $\frac{\theta_5^7\,A^3 + \theta_2^3\,A^2\,B + \theta_{-2}^3\,A\,B^2 + \theta_{-5}^7\,B^3}{2\,A\,B}$. Finally, the second orthology center $[\Gamma_m,\, \Delta_m]$ has $\frac{\theta_{17}^{24}\,A^3 + \theta_1^2\,A^2\,B + \theta_{-1}^2\,A\,B^2 + \theta_{-17}^{24}\,B^3}{2\,A\,B}$

and $\frac{-\theta_7^{10}\,A^3+\theta_3^4\,A^2\,B+\theta_{-3}^4\,A\,B^2-\theta_{-7}^{10}\,B^3}{2\,A\,B}$ as coordinates. The square of the distance between $H(\Gamma_m)$ and $[\Gamma_m,\,\Delta_m]$ is $\theta_{956}^{1352}\,A^6+\theta_{-956}^{1352}\,B^6$ while the square of the distance between the points $H(\Delta_m)$ and $[\Delta_m,\,\Gamma_m]$ is exactly half of this value. □

Theorem 3. *For all positive integers m the oriented areas $|\Delta_m|$ and $|\Gamma_m|$ of the triangles Δ_m and Γ_m are as follows:*

$$|\Delta_m| = 4\,(-1)^m \quad \text{and} \quad |\Gamma_m| = 2\,|\Delta_{m+1}| = 8\,(-1)^{m+1}.$$

Proof. Let us again assume that $\alpha^m = A$ and $B = \beta^m$. Note that $\alpha\,\beta = -1$ so that $A\,B = (-1)^m$. Recall that the triangle with the vertices whose coordinates are $(x_1,\,x_2)$, $(y_1,\,y_2)$, and $(z_1,\,z_2)$ has the oriented area equal to $\frac{(z_1-y_1)x_2+(x_1-z_1)y_2+(y_1-x_1)z_2}{2}$. By direct substitution and simplification we get that $|\Delta_m| = 4\,A\,B = 4\,(-1)^m$. On the other hand, for Γ_m we get $|\Gamma_m| = -8\,A\,B = 2\,|\Delta_{m+1}| = 8\,(-1)^{m+1}$. □

At this point we can go back and keep coordinates of vertices according to their original definition and discover that the first claim in the above theorem is equivalent to the identity

$$P_m\,(P_{m+2}-P_{m+3})+P_{m+1}\,(P_{m+2}+P_{m+3}) = P_{m+1}^2+P_{m+2}^2+8(-1)^m,$$

while the second claim in the above theorem is equivalent to the identity

$$Q_m(Q_{m+2}-Q_{m+3})+Q_{m+1}(Q_{m+2}+Q_{m+3}) = Q_{m+1}^2+Q_{m+2}^2-16(-1)^m.$$

Theorem 4. *For all natural numbers m the centroids $G(\Delta_m)$ and $G(\Gamma_m)$ of the triangles Δ_m and Γ_m are at the distance $\frac{\sqrt{34\,P_{2m}+26\,Q_{2m}}}{3}$.*

Proof. With the notation from the proof of Theorem 2 we get that the centroids $G(\Delta_m)$ and $G(\Gamma_m)$ have $\left(\frac{\theta_5^6\,A+\theta_{-5}^6\,B}{6},\,\frac{\theta_{11}^{16}\,A+\theta_{-11}^{16}\,B}{6}\right)$ and $\left(\frac{\theta_3^5\,A+\theta_{-3}^5\,B}{3},\,\frac{\theta_8^{11}\,A+\theta_{-8}^{11}\,B}{3}\right)$ as coordinates. The square of their distance is $\frac{\theta_{17}^{26}\left(49\,A^2+\theta_{-442}^{627}\,B^2\right)}{441}$ which in turn is precisely $\frac{34\,P_{2m}+26\,Q_{2m}}{9}$. □

The following interesting identity represents an equivalent way in which we can state the above theorem.

$$34\,P_{2m}+26\,Q_{2m}-(P_m-Q_m)^2-(P_{m+3}-Q_{m+3})^2 =$$
$$2\,(P_{m+1}+P_{m+2}-Q_{m+1}-Q_{m+2})(P_m+P_{m+1}+P_{m+2}+P_{m+3}-Q_m-Q_{m+1}-Q_{m+2}-Q_{m+3}).$$

Theorem 5. *For all positive integers m the circumcenters $O(\Delta_m)$ and $O(\Gamma_m)$ of the triangles Δ_m and Γ_m are at the distance $\frac{\sqrt{P_{2m+3}(10+Q_{4m+6})}}{2}$.*

Proof. With the notation from the proof of Theorem 2 we get that the circumcenters $O(\Delta_m)$ and $O(\Gamma_m)$ have

$$\left(\frac{-\theta^{(A+B)(17A^2-22AB+17B^2)}_{4(A-B)(3A^2+2AB+3B^2)}}{4AB}, \frac{\theta^{(A+B)(7A^2+6AB+7B^2)}_{(A-B)(5A^2+14AB+5B^2)}}{4AB} \right)$$

and

$$\left(\frac{\theta^{8(A+B)(3A^2-2AB+3B^2)}_{(A-B)(17A^2+22AB+17B^2)}}{4AB}, \frac{-\theta^{2(A+B)(5A^2-14AB+5B^2)}_{(A-B)(7A^2-6AB+7B^2)}}{4AB} \right)$$

as coordinates. The square of their distance is

$$\frac{\theta^{10(A^2+B^2)(197(A^2+B^2)-186)}_{7(A^2-B^2)(199(A^2+B^2)+210)}}{8}$$

which is equal to

$$\frac{985\, Q_{6m} + 1393\, P_{6m} + 55\, Q_{2m} + 77\, P_{2m}}{4}$$

and therefore to $\frac{P_{2m+3}(10+Q_{4m+6})}{4}$. □

Theorem 6. *For every positive integer m, the triangles Γ_m and Δ_m are reversely similar and the sides of Γ_m are $\sqrt{2}$ times longer than the corresponding sides of Δ_m.*

Proof. It is well-known that two triangles are reversely similar if and only if they are ortologic and paralogic (see [2]). Since, by Theorem 1, we know that triangles Γ_m and Δ_m are orthologic, it remains to see that they are paralogic.

Recall that triangles ABC and XYZ with coordinates of points (a_1, a_2), (b_1, b_2), (c_1, c_2), (x_1, x_2), (y_1, y_2) and (z_1, z_2) are paralogic if and only if the expression $U - V$ is equal to zero where

$$U = \begin{vmatrix} a_1 & b_1 & c_1 \\ x_2 & y_2 & z_2 \\ 1 & 1 & 1 \end{vmatrix}, \qquad V = \begin{vmatrix} a_2 & b_2 & c_2 \\ x_1 & y_1 & z_1 \\ 1 & 1 & 1 \end{vmatrix}.$$

In our situation when we represent coordinates of vertices of triangles Δ_m and Γ_m by the Binet formula in terms of α and β by substitution and easy simplification we get that $U - V = 0$ so that these triangles are indeed paralogic. In a similar way one can easily show that $|X_m Y_m|^2 = 2\,|A_m B_m|^2$. □

Theorem 7. *For every positive integer m, the triangles Γ_m and Δ_m are both orthologic and paralogic. The centers $[\Delta_m, \Gamma_m]$ and $\langle \Delta_m, \Gamma_m \rangle$ are antipodal points on the circumcircle of Δ_m. The centers $[\Gamma_m, \Delta_m]$ and $\langle \Gamma_m, \Delta_m \rangle$ are antipodal points on the circumcircle of Γ_m.*

Proof. The first claim has been established in the previous theorem. In order to prove the second claim we shall prove that the orthology center $[\Delta_m, \Gamma_m]$ lies on the circumcircle of Δ_m by showing that it has the same distance from its circumcenter $O(\Delta_m)$ as the vertex A_m and that the reflection of the point $\langle\Delta_m, \Gamma_m\rangle$ in the circumcenter $O(\Delta_m)$ agrees with the point $[\Delta_m, \Gamma_m]$ (because their distance is equal to zero!).

In the proof of Theorem 5 we found the coordinates of the point $O(\Delta_m)$ and in the proof of Theorem 2 of the center $[\Delta_m, \Gamma_m]$. The coordinates of the center $\langle\Delta_m, \Gamma_m\rangle$ are $\left(\frac{\theta^{4(A+B)}_{3(A-B)}}{2}, \frac{\theta^{10(A+B)}_{7(A-B)}}{2}\right)$. Now it is easy to establish that $|[\Delta_m, \Gamma_m]O(\Delta_m)|^2 - |O(\Delta_m)\, A_m|^2 = 0$. On the other hand, if W denotes the reflection of the point $\langle\Delta_m, \Gamma_m\rangle$ in the circumcenter $O(\Delta_m)$ (i. e., W divides the segment $\langle\Delta_m, \Gamma_m\rangle O(\Delta_m)$ in ratio -2), then $|W[\Delta_m, \Gamma_m]|^2 = 0$.

The third claim has a similar proof. □

Theorem 8. *The square of the diameter of the circumcircle of the triangle Δ_m is equal to $2\,(P_{2m+3})^2\, P_{2m+1}$.*

Proof. In the proof of the previous theorems we found the coordinates of the circumcenter $O(\Delta_m)$. Hence, the square of its distance from the vertex A_m is $169\, Q_{6m} + 31\, Q_{2m} + 239\, P_{6m} + 43\, P_{2m}$. However, this expression is in fact $\frac{P^2_{2m+3}\, P_{2m+1}}{2}$. □

Let k be a positive integer. Let Φ_k and Ψ_k denote the triangles with vertices $D_k = (P_k, Q_{k+1})$, $E_k = (P_{k+1}, Q_{k+2})$, $F_k = (P_{k+2}, Q_{k+3})$ and $U_k = (Q_k, P_{k+1})$, $V_k = (Q_{k+1}, P_{k+2})$, $W_k = (Q_{k+2}, P_{k+3})$, respectively.

In order to describe our next results, recall that triangles ABC and XYZ are *homologic* provided lines AX, BY, and CZ are concurrent. The point P in which they concur is their homology *center* and the line ℓ containing intersections of pairs of lines (BC, YZ), (CA, ZX), and (AB, XY) is their homology *axis*.

In stead of homologic, homology center, and homology axis many authors use the terms *perspective*, *perspector*, and *perspectrix*.

Theorem 9. *For all positive integers m the lines D_mU_{m+1}, E_mV_{m+1} and F_mW_{m+1} are parallel to the line $y = x$ so that the triangles Φ_m and Ψ_{m+1} are homologic. Their homology center is the point at infinity and their homology axis is the line $y = x$. They are never orthologic neither paralogic. The oriented area of both triangles is $4\,(-1)^m$.*

Proof. The lines D_mU_{m+1}, E_mV_{m+1} and F_mW_{m+1} have equations $x - y + \frac{\theta^{2(A+B)}_{A-B}}{2} = 0$, $x - y + \frac{\theta^{4(A+B)}_{3(A-B)}}{2} = 0$, and $x - y + \frac{\theta^{10(A+B)}_{7(A-B)}}{2} = 0$. It follows that they are parallel to the line $y = x$.

Since

$$E_m F_m \cap V_{m+1} W_{m+1} = \left(\frac{\theta^{12AB}_{-8AB}}{\theta^{A+17B}_{-12B}}, \frac{\theta^{12AB}_{-8AB}}{\theta^{A+17B}_{-12B}} \right),$$

$$F_m D_m \cap W_{m+1} U_{m+1} = \left(\frac{\theta^{16AB}_{-12AB}}{\theta^{A-17B}_{12B}}, \frac{\theta^{16AB}_{-12AB}}{\theta^{A-17B}_{12B}} \right),$$

and

$$D_m E_m \cap U_{m+1} V_{m+1} = \left(\frac{\theta^{4AB}_{-4AB}}{\theta^{A-3B}_{2B}}, \frac{\theta^{4AB}_{-4AB}}{\theta^{A-3B}_{2B}} \right),$$

we conclude that the homology axis of the triangles Φ_m and Ψ_{m+1} is the line $y = x$.

The above conditions for the triangles Φ_m and Ψ_{m+1} to be orthologic and paralogic are both equal to $16\,(-1)^{m+1} = 0$ which is not true for any value m. □

The claims about the oriented areas of the triangles Φ_m and Ψ_{m+1} are equivalent to the following identities:

$$2\,Q_{m+2}\,P_{m+1} = (Q_{m+3} - Q_{m+2})P_m + (Q_{m+2} - Q_{m+1})P_{m+2} + 8(-1)^m,$$

$$2\,Q_{m+2}\,P_{m+3} + 8(-1)^m = (Q_{m+3} - Q_{m+2})P_{m+2} + (Q_{m+2} - Q_{m+1})P_{m+4}.$$

Theorem 10. *The triangles Δ_m and Φ_m have equal Brocard angles.*

Proof. It is well-known that the cotangent of the Brocard angle of the triangle with vertices $A(x,\,a)$, $B(y,\,b)$ and $C(z,\,c)$ is equal to

$$\frac{2\begin{vmatrix} x & y & z \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}}{(y-z)^2 + (z-x)^2 + (x-y)^2 + (b-c)^2 + (c-a)^2 + (a-b)^2}.$$

Hence, by direct substitution of coordinates and simplification we discover that the triangles Δ_m and Φ_m both have cotangents of its Brocard angles equal to $\frac{8}{6+(-1)^m(82\,P_{2m}+59\,Q_{2m})}$. □

In a similar way one can show the following result.

Theorem 11. *The cotangent of the Brocard angle of the triangle Ψ_{m+1} is equal to $\frac{8}{6+(-1)^m\,(298\,P_{2m}+211\,Q_{2m})}$.*

Theorem 12. *For all positive integers m the lines $D_m X_{m+1}$, $E_m Y_{m+1}$ and $F_m Z_{m+1}$ are parallel to the line $y = 2\,x$ so that the triangles Φ_m and Γ_{m+1} are homologic. Their homology center is the point at infinity and their homology axis is the line $y = x$. They are never orthologic neither paralogic.*

Proof. The lines $D_m X_{m+1}$, $E_m Y_{m+1}$ and $F_m Z_{m+1}$ have equations $2x - y + A + B = 0$, $2x - y + \theta_{A-B}^{A+B} = 0$, and $2x - y + \theta_{2(A-B)}^{3(A+B)} = 0$. It follows that they are parallel to the line $y = 2x$.

Since $E_m F_m \cap Y_{m+1} Z_{m+1} = E_m F_m \cap V_{m+1} W_{m+1}$, $F_m D_m \cap Z_{m+1} X_{m+1} = F_m D_m \cap W_{m+1} U_{m+1}$, and $D_m E_m \cap X_{m+1} Y_{m+1} = D_m E_m \cap U_{m+1} V_{m+1}$, we conclude that the homology axis of the triangles Φ_m and Γ_{m+1} is the line $y = x$.

The above conditions for the triangles Φ_m and Γ_{m+1} to be orthologic and paralogic are both equal to $24\,(-1)^{m+1} = 0$ which is not true for any value m. □

Theorem 13. *For all positive integers m the triangle Δ_m is orthologic to both triangles Φ_{m+1} and Ψ_{m+1}.*

Proof. The expression $\begin{vmatrix} a_1 & b_1 & c_1 \\ x_1 & y_1 & z_1 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} a_2 & b_2 & c_2 \\ x_2 & y_2 & z_2 \\ 1 & 1 & 1 \end{vmatrix}$ for triangles Δ_m and Φ_{m+1} is equal to $2(\alpha - 1)(\beta - 1)(\alpha^2\,\beta + \alpha\,\beta^2 + 2)\,\alpha^m\,\beta^m$ and therefore to zero because $\alpha\,\beta = -1$ and $\alpha + \beta = 2$. For the triangles Δ_m and Ψ_{m+1} we get that this expression is equal to

$$2(\alpha - 1)(\beta - 1)(2\alpha\,\beta + \alpha + \beta)\,\alpha^m\,\beta^m$$

so that the same conclusion holds. □

In fact, it is possible to prove the following better results:

- For natural numbers m and n the triangles Δ_m and Φ_n are orthologic if and only if $n = m + 1$.
- For natural numbers m and n the triangles Δ_m and Ψ_n are orthologic if and only if $n = m + 1$.

On the other hand, for the triangles Γ_m, we can analogously prove the following results.

- For all natural numbers m and n the triangle Γ_m is not orthologic neither with the triangle Φ_n nor with the triangle Ψ_n.

Theorem 14. *For all positive integers m the triangle Δ_m is paralogic to the triangle Ψ_m. Moreover,*

$$2\,|\langle\Delta_m,\,\Psi_m\rangle\,\langle\Psi_m,\,\Delta_m\rangle|^2 = P_m^2\,P_{m+1}.$$

Proof. The expression $\begin{vmatrix} a_1 & b_1 & c_1 \\ x_2 & y_2 & z_2 \\ 1 & 1 & 1 \end{vmatrix} - \begin{vmatrix} a_2 & b_2 & c_2 \\ x_1 & y_1 & z_1 \\ 1 & 1 & 1 \end{vmatrix}$ for triangles Δ_m and Ψ_m is equal to $2(\alpha - 1)(\beta - 1)(2 - \alpha - \beta)\,\alpha^m\,\beta^m$ and therefore to zero because $\alpha + \beta = 2$.

With the notation from the proof of Theorem 2 we get that the points $\langle \Delta_m, \Psi_m \rangle$ and $\langle \Psi_m, \Delta_m \rangle$ have

$$\left(\frac{-\theta^{(A+B)(3A^2-14AB+3B^2)}_{2(A-B)(A^2-3AB+B^2)}}{4AB}, \frac{\theta^{(B-A)(7A^2-22AB+7B^2)}_{(A+B)(A-5B)(B-5A)}}{4\sqrt{2}AB} \right)$$

and

$$\left(\frac{\theta^{(B-A)(B-3A)(3B-A)}_{-2(A+B)(A^2-6AB+B^2)}}{4\sqrt{2}AB}, \frac{\theta^{(B-A)(B-3A)(3B-A)}_{-2(A+B)(A^2-6AB+B^2)}}{4\sqrt{2}AB} \right)$$

as coordinates. The square of their distance is

$$\frac{(A-B)^4\, \theta^{3A^2+2AB+3B^2}_{2(A^2-B^2)}}{32}$$

which is equal to $\frac{P_m^2\, P_{m+1}}{2}$. □

Of course, again it is possible to prove the following better result:

- For natural numbers m and n the triangles Δ_m and Ψ_n are paralogic if and only if $n = m$.

On the other hand, we can also prove the following results.

- For all natural numbers m and n the triangle Δ_m is not paralogic with the triangle Φ_n.
- For all natural numbers m and n the triangle Γ_m is not paralogic neither with the triangle Φ_n nor with the triangle Ψ_n.

In closing, let us observe that the following are also pairs of homologic triangles: (Δ_m, Φ_m), (Δ_m, Ψ_m), (Γ_m, Φ_m) and (Γ_m, Ψ_m). The reason in these cases is quite simple – the corresponding vertices have identical either first or second coordinates.

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INTERSECTIONS OF A "SMALL" CONVEX BODY IN A PLANE LATTICE

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Abstract

Let R_a and $R_{a,\alpha} = R_a \cup R_\alpha$ be the lattice of Buffon respectively of Laplace. For a "small" convex body T (a regular hexagon of constant side) placed at random on the Euclidean plane E_2, we give formulas for

– the probability p_s that T intercepts a line-segment of length at least equal to s on a line of the lattice R_a.

– the probability $p_{s\wedge\sigma}$ (respectively $p_{s\vee\sigma}$) that T intercepts a line-segment of length at least equal to s on a line of the lattice R_a and (respectively or) a line-segment of length at least equal to σ on a line of the lattice R_α.

AMS 2000 Subject Classification : Geometric probability, stochastic geometry, random sets, random convex sets and integral geometry.
AMS Classification : 60D05, 52A22

§ **1**. The Buffon-Laplace problem of calculating the probability that a "small" convex body T in the Euclidean plane E_2 intercepts a line-segment of length at least equal to s on a line of the lattice R_a (strips of constant width) and the probability that T intercepts on $R_{a,\alpha}$ (rectangles with constant sides) a line-segment of length at least equal to s on a line of the lattice R_a and (respectively or) a line-segment of length at least equal to σ on a line of the lattice R_α, has been studied (in [3]) by M. Pettineo, if T is a square or a rectangle.

The purpose of this paper is to discuss analogous problems using a "small" regular hexagon of constant side placed at random on the Euclidean plane E_2 as test body.

In the Euclidean plane E_2 let R_a be a lattice of parallel equidistant lines with distance a.
Let $R_{a,\alpha}$ be a lattice of rectangles $R_a \cup R_\alpha$, where R_α is also a lattice of parallel lines with distance α apart making an angle $\frac{\pi}{2}$ with the lines of R_a.
Let T (a regular hexagon of constant side r) be a random test body "small" (in the sense given in [1]) compared to the lattice R_a and to the lattice $R_{a,\alpha}$, so $2r < \min\{a, \alpha\}$.
For any fixed $s \leq 2r$ we denote by p_s the probability that T intercepts a line-segment of length at least equal to s on a line of the lattice R_a.
For any fixed s, σ (nonnegative real numbers) we denote by $p_{s\wedge\sigma}$ (respectively $p_{s\vee\sigma}$) the probability that T intercepts a line-segment of length at least equal to s on a line of the lattice R_a and (respectively or) a line-segment of length at least equal to σ on a line of the lattice R_α.
We observe that $p_{s\vee\sigma} > 0$ for all $(s, \sigma) \in [0, 2r[\times [0, 2r[$; while $p_{s\wedge\sigma} > 0$, if and only if $(s, \sigma) \in \mathcal{Z}$, whereas the admissible set $\mathcal{Z}$ is

$$\mathcal{Z} = \{(s, \sigma) : s \in [0, 2r[\ ,\ \sigma \in [0, \rho(s)[\} \cup \{(s, \sigma) : s \in [0, \rho(\sigma)[\ ,\ \sigma \in [0, 2r[\}\ .$$

We can omit determining the function $\rho : [0, 2r[\to]\sqrt{3}\, r,\ 2r[$, because it does not play an essential role in the present paper.

§ **2**. Let F be a strip of constant width $\frac{a}{2}$ such that one distinguished straight line g of the lattice R_a is one of the two connected components of the border of F. We take F as the elementary tile of R_a.

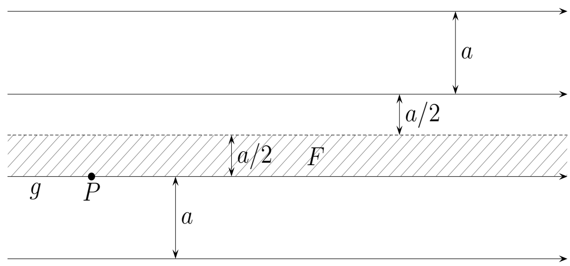

figure 1

We assume the straight lines of R_a and the hexagon T to be oriented as shown in figures 1, 2.

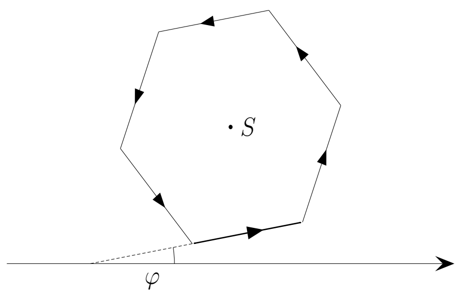

figure 2

If the barycentre S of T is fixed, we obtain, for symmetry reasons, all positions of T with respect to R_a exactly once, if the angle φ between one distinguished oriented side of T and the direction of g varies between 0 and $\frac{\pi}{6}$. To simplify matters we call φ the angle of T.

Let P be a fixed point of g and $\mathcal{M}$ the set of all hexagons T whose barycentres S are in F such that the line-segment $\overline{SP}$ is orthogonal to g. Let from now on also $\varphi \in \left[0, \frac{\pi}{6}\right]$ be fixed.

If $s \in \left[0, \frac{r\sqrt{3}}{\sin(\frac{\pi}{3}+\varphi)}\right)$, there exists a unique hexagon $T_0 \in \mathcal{M}$ with angle φ which intercepts on g a line-segment of length s. Let $x(\varphi)$ be the distance between the barycentre S_0 of T_0 and the line g. We observe that every hexagon T of $\mathcal{M}$ with angle φ intercepts a line-segment of length at least equal to s on a line g if and only if its barycentre S is nearer to g than S_0.

If $s = \frac{r\sqrt{3}}{\sin(\frac{\pi}{3}+\varphi)}$, every hexagon T of $\mathcal{M}$ with angle φ such that the distance of the barycentre S of T to g is at least equal to $r\sin\varphi$ intercepts a line-segment of length s on g. We define $x(\varphi) = r\sin\varphi$.

If $s > \frac{r\sqrt{3}}{\sin(\frac{\pi}{3}+\varphi)}$, each hexagon T of $\mathcal{M}$ with angle φ intercepts a line-segment of length smaller than s on g, and therefore we put $x(\varphi) = 0$.

We use a well-known Stoka's formula (see [4]) in order to compute the probability p_s and get

$$p_s = \frac{\int_0^{\frac{\pi}{6}} x(\varphi)d\varphi}{\int_0^{\frac{\pi}{6}} \frac{a}{2}\,d\varphi} = \frac{\int_0^{\frac{\pi}{6}} x(\varphi)d\varphi}{\frac{a\pi}{12}} = \frac{12}{\pi a}\int_0^{\frac{\pi}{6}} x(\varphi)d\varphi \,. \tag{1}$$

We have to consider three cases in order to compute $x(\varphi): 0 \le s \le r\,,\ r \le s \le r\sqrt{3}\,,\, r\sqrt{3} \le s \le 2r\,.$

If $0 \le s \le r$, we have (see figure 3)

$$x(\varphi) = r \sin\left(\frac{\pi}{3} + \varphi\right) - \frac{s}{\sqrt{3}}\left[\cos\left(\frac{\pi}{3} - 2\varphi\right) - \frac{1}{2}\right]\,. \tag{2}$$

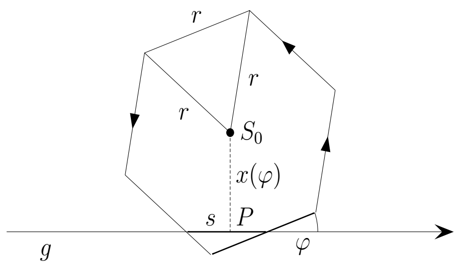

figure 3

Therefore

$$\int_0^{\frac{\pi}{6}} x(\varphi)d\varphi \;=\; \frac{r}{2} - \frac{s}{4}\left[1 - \frac{\pi}{3\sqrt{3}}\right]\,. \tag{3}$$

If $r \le s \le \sqrt{3}\,r$, let $\varphi_0 \in \left[0, \frac{\pi}{6}\right]$ be such that $\sin\left(\frac{\pi}{3} - \varphi_0\right) = \frac{r\sqrt{3}}{2s}\,.$

If $\varphi \in [0, \varphi_0]$, we have (see figure 4)

$$x(\varphi) \;=\; \sqrt{3}\,r\cos\varphi - \frac{s}{\sqrt{3}}\left[\cos(2\varphi) + \frac{1}{2}\right]\,. \tag{4}$$

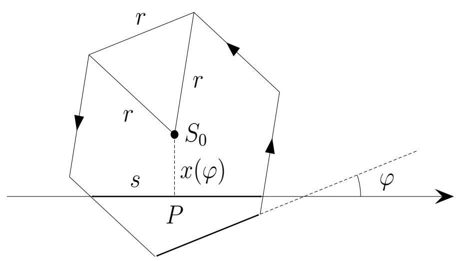

figure 4

If $\varphi \in [\varphi_0\,, \frac{\pi}{6}]$, we obtain again formula (3) for $x(\varphi)$.
Thus

$$\begin{aligned}
&\int_0^{\frac{\pi}{6}} x(\varphi)d\varphi \;=\; \int_0^{\varphi_0} x(\varphi)d\varphi + \int_{\varphi_0}^{\frac{\pi}{6}} x(\varphi)d\varphi \;=\; \sqrt{3}\,r\sin\varphi_0 + r\cos\left(\frac{\pi}{3} + \varphi_0\right) \\
(5)\qquad &-\frac{1}{2\sqrt{3}}\sin 2\varphi_0 - \frac{s}{2\sqrt{3}}\sin\left(\frac{\pi}{3} - 2\varphi_0\right) + \frac{2}{2\sqrt{3}}\left(\frac{\pi}{6} - 2\varphi_0\right) = \\
&r\sin\left(\varphi_0 + \frac{\pi}{6}\right) - \frac{s}{2\sqrt{3}}\cos\left(\frac{\pi}{6} - 2\varphi_0\right) + \frac{s}{2\sqrt{3}}\left(\frac{\pi}{6} - 2\varphi_0\right)\,.
\end{aligned}$$

Finally if $r\sqrt{3} \le s \le 2r$, let $\varphi_1 \in \left[0, \frac{\pi}{6}\right]$ be such that $\sin\left(\varphi_1 + \frac{\pi}{3}\right) = \frac{r\sqrt{3}}{2}$.
If $\varphi \in \left(\varphi_1, \frac{\pi}{6}\right]$, we have $x(\varphi) = 0$, because the test body T intercepts a line-segment of length smaller than s on g; if $\varphi \in [0, \varphi_1]$, we use for $x(\varphi)$ the formula (4). Therefore

$$\int_0^{\frac{\pi}{6}} x(\varphi)d\varphi = \int_0^{\varphi_1} x(\varphi)d\varphi = \sqrt{3}\,r\sin\varphi_1 - \frac{s}{2\sqrt{3}}\left(\varphi_1 + \sin 2\varphi_1\right). \tag{6}$$

Formulas (1), (3), (5) and (6) lead to the following proposition.

Proposition 1. The probability p_0 that a regular hexagon (with a constant side r) placed at random on the Euclidean plane E_2 intercepts a line-segment of length at least equal to s (where $s \le 2r < a$) on a line of the lattice R_a is given by

$$p_s = \frac{6r}{\pi a} - \frac{s}{a}\left(\frac{3}{\pi} - \frac{1}{\sqrt{3}}\right) \quad \text{if } 0 \le s \le r, \tag{7}$$

$$p_s = \frac{12}{\pi a}\left[r\sin\left(\varphi_0 + \frac{\pi}{6}\right) + \frac{s}{2\sqrt{3}}\left(\frac{\pi}{6} - 2\varphi_0\right) - \frac{s}{2\sqrt{3}}\cos\left(\frac{\pi}{6} - 2\varphi_0\right)\right] \quad \text{if } r \le s \le r\sqrt{3}, \tag{8}$$

$$p_s = \frac{12}{\pi a}\left[\sqrt{3}\,r\sin\varphi_1 - \frac{s}{2\sqrt{3}}(\varphi_1 + \sin 2\varphi_1)\right] \quad \text{if } r\sqrt{3} \le s \le 2r, \tag{9}$$

whereas the angles $\varphi_0, \varphi_1 \in \left[0, \frac{\pi}{6}\right]$ are uniquely determined by the conditions $\sin\left(\frac{\pi}{3} - \varphi_0\right) = \frac{r\sqrt{3}}{2s}$ respectively $\sin\left(\varphi_1 + \frac{\pi}{3}\right) = \frac{r\sqrt{3}}{s}$.

Remarks:

a) If $s = 0$, the probability p_s (that a random regular hexagon of constant side r with the condition $2r < a$ intersects the lattice R_a) becomes $p_0 = \frac{6r}{\pi a}$. This probability has been computed by M.I. Stoka ([4]).

b) If $s = 2r$, we have $\varphi_1 = 0$ and $p_{2r} = 0$.

c) p_s is a continuous function of s:
– if $s = r$, taking account of (7) and (8), we have the same result $p_r = \left(\frac{3}{\pi} + \frac{1}{\sqrt{3}}\right)\frac{r}{a}$;
– if $s = r\sqrt{3}$, taking account of (8) and (9), we have the samt result $p_{r\sqrt{3}} = \left(\frac{3\sqrt{3}}{\pi} - 1\right)\frac{r}{a}$.

d) If we consider the lines of the lattice R_a like pipe-conduit (of arbitrary nature), then the number $p_s - p_t (s < t < 2r)$ gives the probability that the random hexagon T causes a damage bigger than s and smaller than t.
This result may be used to fix premiums of insurance.

e) The analysis of the above-mentioned hints on a general technique to compute the probabiltiy that a regular polygon T_n with n $(n \ge 5)$ sides of constant length r intercepts a line-segment of length at least equal to s $(s \le \text{diam}T_n \le a)$ on a line of the lattice R_a.

The case of an equilateral triangle is substantially different and has been investigated by A. Duma and M.I. Stoka ([2]).

§ 3. Now we replace the above considered lattice R_a of parallel lines with equidistance a by a lattice of rectangles $R_{a,\alpha} = R_a \cup R_\alpha$, where R_α is also a lattice of parallel lines with distance α including an angle $\frac{\pi}{2}$ with the lines of R_a . We denote by $\mathcal{C}$ the fundamental cell of $R_{a,\alpha}$; hence $\mathcal{C}$ is a rectangle of sides a and α .

Our purpose is now to compute the probabilies $p_{s\wedge\sigma}$ and $p_{s\vee\sigma}$ for $\max\{s,\sigma\} \le r$, because in this case all pairs (s,σ) are admissible, and because we avoid the consideration of different cases.

Let the border of T and $\mathcal{C}$ be positively oriented (anticlockwise). Let φ be the angle between the lines of R_a and the marked side of the hexagon T; in short φ is the angle of T. To obtain all the possible positions of T with respect to $R_{a,\alpha}$ (if the barycentre S of T is fixed) it is enough to consider $\varphi \in \left[0, \frac{\pi}{6}\right]$, because another side of T will shape the angle $\frac{\pi}{6} - \varphi$ with the lines of R_α .

If T with angle φ intercepts a line-segment of length s (respectively σ) on a line of R_a (respectively R_α), let $x_s(\varphi)$ (respectively $y_\sigma(\varphi)$) be the distance of the barycentre S_0 of T to the corresponding side of R_a (respectively R_α).

It follows that $x_s(\varphi)$ is given by formula (2) and, because $y_\sigma(\varphi) = x_\sigma\left(\frac{\pi}{6} - \varphi\right)$, we have

$$y_\sigma(\varphi) = r\cos\varphi - \frac{\sigma}{\sqrt{3}}\left[\cos 2\varphi - \frac{1}{2}\right] . \tag{10}$$

Let $\mathcal{C}_\wedge(\varphi)$ (respectively $\mathcal{C}_\vee(\varphi)$) be the set of points of $\mathcal{C}$ which are nearer to a straight line of R_a and (respectively or) to a straight line of R_α than S_0 .
It is not difficult to prove (the technique is the same as to obtain formula (1)) that

$$p_{s\vee\sigma} = \frac{\int_0^{\pi/6} \text{Area}\,\mathcal{C}_\vee(\varphi)d\varphi}{\int_0^{\pi/6} \text{Area}\,\mathcal{C}d\varphi} = \frac{6}{\pi a\alpha}\int_0^{\pi/6} \text{Area}\,\mathcal{C}_\vee(\varphi)d\varphi \ , \tag{11}$$

$$p_{s\wedge\sigma} = \frac{\int_0^{\pi/6} \text{Area}\,\mathcal{C}_\wedge(\varphi)d\varphi}{\int_0^{\pi/6} \text{Area}\,\mathcal{C}d\varphi} = \frac{6}{\pi a\alpha}\int_0^{\pi/6} \text{Area}\,\mathcal{C}_\wedge(\varphi)d\varphi \ . \tag{12}$$

To calculate Area $\mathcal{C}_\vee(\varphi)$ (respectively Area $\mathcal{C}_\wedge(\varphi)$) we refer to the following fig. 5 (resp. fig. 6).

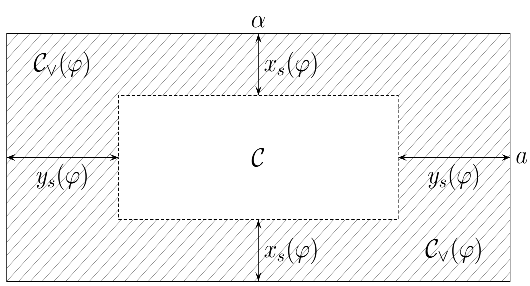

figure 5

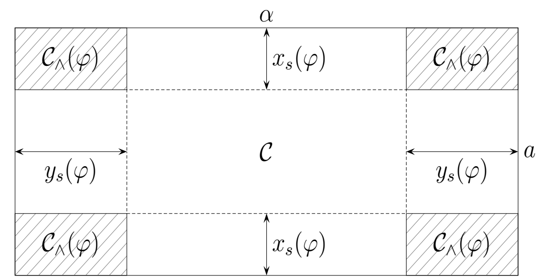

figure 6

It follows

$$\text{(13)}\qquad \operatorname{Area}\mathcal{C}_\vee(\varphi) = 2\alpha x_s(\varphi) + 2[a - 2x_s(\varphi)]y_\sigma(\varphi) = 2\alpha x_s(\varphi) + 2a x_\sigma\left(\frac{\pi}{6} - \varphi\right) - 4x_s(\varphi)x_\sigma\left(\frac{\pi}{6} - \varphi\right),$$

$$\text{(14)}\qquad \operatorname{Area}\mathcal{C}_\wedge(\varphi) = 4x_s(\varphi)y_\sigma(\varphi) = 4x_s(\varphi)x_\sigma\left(\frac{\pi}{6} - \varphi\right).$$

Because

$$\begin{aligned}
&x_s(\varphi)y_\sigma(\varphi) = r^2\sin\left(\frac{\pi}{3} + \varphi\right)\cos\varphi - \frac{r\sigma}{\sqrt{3}}\sin\left(\frac{\pi}{3} + \varphi\right)\cos 2\varphi \\
&-\frac{rs}{\sqrt{3}}\cos\left(\frac{\pi}{3} - 2\varphi\right)\cos\varphi + \frac{s\sigma}{3}\cos\left(\frac{\pi}{3} - 2\varphi\right)\cos 2\varphi + \frac{s\sigma}{12} \\
&-\frac{s\sigma}{6}\left[\cos\left(\frac{\pi}{3} - 2\varphi\right) + \cos 2\varphi\right] + \frac{r}{2\sqrt{3}}\left[\sigma\sin\left(\frac{\pi}{3} + \varphi\right) + s\cos\varphi\right] = \\
&r^2\left[\frac{1}{2}\sin\left(\frac{\pi}{3} + 2\varphi\right) + \frac{\sqrt{3}}{4}\right] - \frac{r\sigma}{2\sqrt{3}}\left[\sin\left(\frac{\pi}{3} + 3\varphi\right) + \sin\left(\frac{\pi}{3} - \varphi\right)\right] \\
&-\frac{rs}{2\sqrt{3}}\left[\cos\left(\frac{\pi}{3} - 3\varphi\right) + \cos\left(\frac{\pi}{3} - \varphi\right)\right] + \frac{s\sigma}{6}\left[\cos\left(\frac{\pi}{3} - 4\varphi\right) + \frac{1}{2}\right] + \frac{s\sigma}{12} \\
&-\frac{s\sigma}{6}\left[\cos\left(\frac{\pi}{3} - 2\varphi\right) + \cos 2\varphi\right] + \frac{r\sigma}{2\sqrt{3}}\sin\left(\frac{\pi}{3} + \varphi\right) + \frac{rs}{2\sqrt{3}}\cos\varphi\,,
\end{aligned}$$

we have

$$\text{(15)}\qquad \begin{aligned}
&\int_0^{\pi/6} x_s(\varphi)y_\sigma(\varphi)d\varphi = \int_0^{\pi/6} x_s(\varphi)x_\sigma\Big(\frac{\pi}{6} - \varphi\Big)d\varphi = \\
&\frac{r^2}{4}\Big(1 + \frac{\pi\sqrt{3}}{6}\Big) - \Big(\frac{1}{3} - \frac{5}{12\sqrt{3}}\Big)(s + \sigma)r + s\sigma\Big(\frac{\pi}{6} + \frac{1}{48} - \frac{\sqrt{3}}{16}\Big)\,.
\end{aligned}$$

The formulas (13), (14) and (15) lead to the following proposition.

Proposition 2. Let $R_{a,\alpha} = R_a \cup R_\alpha$ be a lattice of rectangles in the Euclidean plane E_2. The probability $p_{s\wedge\sigma}$ (respectively $p_{s\vee\sigma}$) that a random regular hexagon T (of constant side r) uniformly distributed in a bounded region of the plane intercepts a line-segment of length at least equal to s on a line of R_a and (respectively or) a line-segment of length at least equal to σ on a line of R_α is

$$p_{s\wedge\sigma} = \frac{r^2(6+\pi\sqrt{3}) - (s+\sigma)r\left(8-\frac{10}{\sqrt{3}}\right) + s\sigma\left(4\pi+\frac{1}{2}-\frac{3\sqrt{3}}{2}\right)}{\pi a\alpha} \tag{16}$$

resp.

$$p_{s\vee\sigma} = \frac{r-3s+\frac{\pi s}{\sqrt{3}}}{\pi a} + \frac{r-3\sigma+\frac{\pi\sigma}{\sqrt{3}}}{\pi\alpha} - \frac{r^2(6+\pi\sqrt{3}) - (s+\sigma)r\left(8-\frac{10}{\sqrt{3}}\right) + s\sigma\left(4\pi+\frac{1}{2}-\frac{3\sqrt{3}}{2}\right)}{\pi a\alpha}, \tag{17}$$

if the conditions $\max\{s,\sigma\} \le r$, $2r \le \min\{a,\alpha\}$ are satisfied.

§ 4. Let $F : [0,2r] \longrightarrow [0,1]$ be the chord-length distribution in the regular hexagon of side r. If $\overline{F} := 1-F$, then with the notations of paragraph 2 follows

$$\overline{F}(s) = p_s/p_0 = \frac{12}{\pi a}\int_0^{\pi/6} x(\varphi)d\varphi \Big/ \frac{6r}{\pi a} = \frac{2}{r}\int_0^{\pi/6} x(\varphi)d\varphi , \tag{18}$$

and hence

$$\overline{F}(s) = 1 - \frac{s}{2r}\left(1-\frac{\pi}{3\sqrt{3}}\right) \ \text{ if } \ 0 \le s \le r , \tag{19}$$

$$\overline{F}(s) = 2\sin\left(\varphi_0+\frac{\pi}{6}\right) + \frac{s}{r\sqrt{3}}\left[\left(\frac{\pi}{6}-2\varphi_0\right) - \cos\left(\frac{\pi}{6}-2\varphi_0\right)\right] \ \text{ if } \ r \le s \le r\sqrt{3} , \tag{20}$$

$$\overline{F}(s) = 2\sqrt{3}\sin\varphi_1 - \frac{s}{r\sqrt{3}}\left[\varphi_1+\sin 2\varphi_1\right] \ \text{ if } \ r\sqrt{3} \le s \le 2r . \tag{21}$$

The density of (the chord-length) distribution f is well-known, if F is differentiable in s, as

$$f(s) \ = \ \frac{dF}{ds}(s) \ = -\frac{d\overline{F}}{ds}(s) .$$

For $s < r$ the density is constant, namely $\frac{1}{2r}\left[1-\frac{\pi}{3\sqrt{3}}\right]$.

For $r < s < r\sqrt{3}$ we have

$$\begin{aligned} f(s) \ &= \ -\left[2\cos\left(\varphi_0+\frac{\pi}{6}\right) - \frac{2s}{r\sqrt{3}} - \frac{2s}{r\sqrt{3}}\sin\left(\frac{\pi}{6}-2\varphi_0\right)\right]\cdot\varphi_0'(s) - \frac{1}{r\sqrt{3}}\left[\left(\frac{\pi}{6}-2\varphi_0\right) - \cos\left(\frac{\pi}{6}-2\varphi_0\right)\right] \\ &= \ \frac{1}{s\cos\left(\frac{\pi}{6}-\varphi_0\right)} + \frac{\sin\left(\frac{\pi}{6}-2\varphi_0\right)}{s\cos\left(\frac{\pi}{6}-\varphi_0\right)} - \frac{\sqrt{3}\,r\cos\left(\varphi_0+\frac{\pi}{6}\right)}{s^2\cos\left(\frac{\pi}{6}-\varphi_0\right)} - \frac{1}{r\sqrt{3}}\left[\left(\frac{\pi}{6}-2\varphi_0\right) - \cos\left(\frac{\pi}{6}-2\varphi_0\right)\right] , \end{aligned}$$

because $\varphi_0'(s) = \dfrac{r\sqrt{3}}{2s^2\cos\left(\frac{\pi}{3}-\varphi_0\right)}$.

We have $\varphi_0 \longrightarrow 0$ for $s \longrightarrow r+$ and

$$\lim_{s\to r+} f(s) = \frac{1}{r}\cdot\frac{2}{\sqrt{3}} + \frac{1}{r}\cdot\frac{1}{\sqrt{3}} - \frac{\sqrt{3}}{r} - \frac{1}{r\sqrt{3}}\left(\frac{\pi}{6}-\frac{\sqrt{3}}{2}\right) \ = \ \frac{1}{2r}\left(1-\frac{\pi}{3\sqrt{3}}\right) \ = \ \lim_{s\to r-} f(s) .$$

For $r\sqrt{3} < s < 2r$ we obtain

$$\begin{aligned} f(s) &= \left[-2\sqrt{3}\cos\varphi_1 + \frac{s}{r\sqrt{3}}(1+2\cos 2\varphi_1)\right]\varphi_1'(s) + \frac{1}{r\sqrt{3}}\left(\varphi_1 + \sin 2\varphi_1\right) \\ &= \frac{2\sqrt{3}\cos\varphi_1 \sin\left(\frac{\pi}{3}+\varphi_1\right)}{s\cos\left(\frac{\pi}{3}+\varphi_1\right)} - \frac{1+2\cos 2\varphi_1}{s\cos\left(\frac{\pi}{3}+\varphi_1\right)} + \frac{1}{r\sqrt{3}}\left(\varphi_1 + \sin 2\varphi_1\right), \end{aligned}$$

because $\varphi_1'(s) = -\dfrac{r\sqrt{3}}{s^2\cos\left(\frac{\pi}{3}+\varphi_1\right)} = -\dfrac{\sin\left(\frac{\pi}{3}+\varphi_1\right)}{s\cos\left(\frac{\pi}{3}+\varphi_1\right)}$.

If $s \longrightarrow r\sqrt{3}$, then $\varphi_0 \longrightarrow \frac{\pi}{6}$ and $\varphi_1 \longrightarrow \frac{\pi}{6}$. Hence

$$\lim_{s\to r\sqrt{3}-} f(s) = \frac{1}{r\sqrt{3}} - \frac{1}{2r\sqrt{3}} - \frac{\sqrt{3}\,r}{2\cdot 3r^2} - \frac{1}{r\sqrt{3}}\left(-\frac{\pi}{6} - \frac{\sqrt{3}}{2}\right) = \frac{1}{2r}\left(1 + \frac{\pi}{3\sqrt{3}}\right),$$

$$\lim_{s\to r\sqrt{3}+} f(s) = \frac{1}{r\sqrt{3}} \lim_{\varphi_1\to\frac{\pi}{6}} \frac{2\sqrt{3}\cos\varphi_1\sin\left(\frac{\pi}{3}+\varphi_1\right) - 1 - 2\cos 2\varphi_1}{\cos\left(\frac{\pi}{3}+\varphi_1\right)} + \frac{1}{2r}\left(1+\frac{\pi}{3\sqrt{3}}\right).$$

Because $\lim\limits_{\varphi_1\to\frac{\pi}{6}} \cos\left(\dfrac{\pi}{3}+\varphi_1\right) = 0$ and

$$\lim_{\varphi_1\to\frac{\pi}{6}} \left[2\sqrt{3}\cos\varphi_1\sin\left(\frac{\pi}{3}+\varphi_1\right) - 1 - 2\cos 2\varphi_1\right] = 2\sqrt{3}\cdot\frac{\sqrt{3}}{2}\cdot 1 - 1 - 2\cdot\frac{1}{2} = 1$$

the distribution function F is not differentiable in $s = r\sqrt{3}$.

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STRATEGIES FOR THE DEVELOPMENT OF INTERMODAL TRANSPORT INFRASTRUCTURES: THE PROPOSALS OF REGION SICILY IN EUROPEAN RE.MO.MED PROJECT

FERDINANDO CORRIERE – PIETRO AIELLO

Abstract. This paper is a short report of ReMoMed (Réseau européen intermodal pour un développement intégré des espaces de la Méditerranée occidentale) project's executive summary, so as it has been pointed out by Department of City and Territory of Palermo University and executed by Sicily Region.
In particular, at first, a synthesis of the methodological approach of the executive project phases and then the Pilot action and the experimentation of the model, based on the development objectives of the Augusta harbour, has been represented.
All, obviously, is related whit the state of infrastructures in Sicily, above all, whit their efficiency and economic management, and whit their prospective of development.

Keywords: infrastructures, management, integrated development

1. Introduction

The harbour infrastructures of the Region allow to restore the territorial continuity with the entire peninsula and with the smaller Islands.

The marine freight transport in Sicily is important regarding the national freight traffic because it approximately absorbs its 41.3%, and has always crescent trend both for the difficulties of realization of land infrastructures connected to other transport systems, and for the increase of freight transport demand in the natural corridor constituted by the Mediterranean Sea (to reach the markets of the North Italy, of North Europe and of the other countries in the Mediterranean basin).

The total number of approaches in the main Sicilian ports amounts to 134 of which 124 ones are mostly dedicates to freight transport.

There are 17 available approaches for the Ro-Ro services while the remaining part is dedicated to sport sailing, to fishing, to services and to the moorings of military ships.

Although the appearing good equipment of approaches, those to be renovated comes to 70 (approximately 32% of the national amount). In spite of the inadequacy of economic investments, in the last years connections from the main Sicilian ports towards Italians port terminals is developed. Recently in fact, new lines of marine transport have been instituted (above all of Roll On-Roll Off type), these lines involve the ports of Palermo and Catania towards the Tyrrhenian, Liguria and Adriatic coasts with an average frequency of, approximately, three race a week.

Anyway, the absence of an effective unit logistic coordination of the Sicilian port system, whose management is not facilitated by the elevated number of Harbour Authorities in the Island. Moreover the position of main port terminals inside the city centres contributes to increase the traffic's congestion of the same cities and compromises accessibility to other main land transport infrastructures, especially with regard to the bad distribution of the railway connections (over dimensioned in a few ports and totally absent in others).

Confronting the infrastructure equipments of each Sicilian ports with the Italian average, it appears paradoxical an excess of approaches and mechanical furnishing in spite of the insufficiency of goods large squares, of a diffuse deficiency of silos and warehouses, above all those for the cold.

The main critical points in the marine traffic, as well evidenced in the "Sicily Region Transport Master Plan", are so summarized:

- Lack in equipments and land spaces;
- Lack in infrastructures connections with land transport nets and consequent problems of accessibility;
- Inadequacy of depth;
- Inadequacy of terminal equipments for the containers traffic;
- Lack of specialist equipments for the ro-ro traffic, such as gating systems and for the weighing of vehicles;
- Lack of equipment of advanced systems for the marine traffic control, also in order to safety;
- Inadequacy of the logistic chains.

From the data of the statistical "Yearbook 2001 ", edited by the "Ministry of Transport an Infrastructures" and by "E.N.A.C.", it is possible to summarize the volumes of traffic managed from th regional ports of call.

In 2001 commercial traffic monitored in Italy has been equal to 90.21 0.038 passenger and 723.,00 ton of goods; Sicily, with its 7.645.31 5 passenger and 1 1 .693 ton of goods (non oil) represents only 8.5 and 1 .6% of the national amount respectively.

It means that the Region, in spite of its barycentre strategic position in the Mediterranean basin, in th both national and foreign rout rows is not considered.

Catania Fontanarossa comes out as the call with the greater handling of goods, and as the one for which the trend of increase is more marked and constant. The Punta Raisi airport instead seems to hav caught up one saturation stage. The airport calls of Pantelleria and Lampedusa have a very low traffic and mostly under seasonal factor, but at the same time, they play a fundamental role in the accessibility to the islands. The Trapani Birgi airport is currently overdimensioned and so with an high potential available.

In the critical point analysis has been included:

- The analysis of the normative framework in which has developed the actual intermodal system of distribution of goods between Sicilian Region and southern area of the Mediterranean;
- The recognition of the critical points in the normative framework that regulates the performanc of the intermodal system in object.

Notably in 1 980, in the framework of the United Nations, the Genève Convention on the intermoda freight transport was adopted. But such agreement has never entered in force, consequently, it does n involves binding legal effects. In 1 992, instead, in the framework of the Conference of the United Nations on the Commerce and Development (UNCTAD), was adopted a contractual standard rules se incorporable in the trade contracts of that the private parts.

On the base of the "multilevel" normative framework (international, communitarian, national an regional) have been synthetically summarized only some of the problem profiles that should regard the project in examination.

. Coordination beteen decisional competences

After the reform of Title V of the Constitution, in our legal regulations the transport sector has becom of concurrent legal power and, for some aspects, of regional exclusive legal competence. Such situation places the regions in a position of greater decisional autonomy, but also in the condition to find direct coordination not only with the others, equal former Italian regional authorities, but above all with the competent decisional levels of the sector in the other Countries interested in the development of Projec RE.MO.MED. In this field, it's to highlight that the communitarian norm (the programs "Marco Polo" and TEN-T) privileges the financial boosting and support of project of wide range (the s.c. "intermoda corridors") which involved more Member States (MS) of the European Union.

. Pub/ics aids in intermodality deelopment: compatibi/ity conditions with the communitarian ru/es

The praxis of the European Commission as regard the control of lawfulness of the proposals of aids required by MS, brings to light which conditions is necessary to satisfy when a public body desires to engage itself in the financial support of intermodal transport. According to the art. 87,1 TCE, is incompatible with the common market every aid given from the bodies of one MS or with the resources of one MS, in case that it touch or threatens to affect the exchanges between MS, distorting or threatening to distort the free game of the competition in favour of same enterprises or productions.

Therefore, the aids given to operators of the combined transport, in case that they give them an advantage trade with public resources which other operators do not enjoy (i.e. reducing the sector tra barriers), re-enter in the notion of prohibited public aid provided by the art. 87,1 TCE . But the incompatible aid can be one of the exceptions at the prohibition of publics aids contemplated from the communitarian Treaty.

Notably, between the provided exceptions, there is that one established by the art. 73 TCE for the aids required for the necessity of the coordination of the transports in fields in which there isn't a competitive market or with market imperfections.

Also there is that one established by the art. 87,3(c) for the aids considerable compatible with the common market as assigned to facilitate the development of some economic activities, if they don't alter the exchanges conditions against the common interest. Of great importance for the transport discipline is the Environmental Protection issue. Many and detailed measures have been adopted, i.e. in communitarian framework, in order to prevent or to limit the environmentally damaging consequences from of the different modes of transport (notably road, marine and aerial). Between them, important are the measures on the standards of emission. As regard, notably, the combined transport, the program Marco Polo has like specific scope just that one "to move" the freight transport towards modes of transport alternatives to the road, that could express elevated environmental performances.

1.3 Safety and protection

It enters in the competences of the Council of the European Union according to the art. 71 ,1 (c) TCE, introduced with the Treaty of Mastricht. In a future perspective, the White Book of 2001 on European transport policy, preannounce legislative initiatives of the Commission directed to improve safety of the tunnels included in the trans-European net. More and more pressing, then, is the tentative to create systems and services of "intelligent transport", directed not only to improve the safety of the several types of transport, but also to reduce the transport costs and environmental impact

2. Methodological Approach

The phase's aim is the "definition of the action strategies for the development of the Euro-Mediterranean intermodal network for the freight transport". The objective is to lay fundaments of a regional strategic plan to articulate various top-level measures to develop the intermodality of logistics hubs in the Sicilian Region.

The methodology consists of several phases:

- A SWOT (Strengths-Weaknesses -Opportunities-Threat) analysis starting from the results of the previous phases of the project;
- Analysis of the Swot's results with stakeholders;
- Definition of a priority scale of this results;
- Definition of measures for main elements, which have to satisfy some characteristics;
- Priority;
- Economic aids;
- Prevision of measures for accelerating procedures;
- Prevision of measures to be implemented at European level;
- Prevision of measures for infrastructures;
- Classification of these measures and grouping in some action lines to implement it in a structural plan.

The methodology proposes a controlling model of action lines through the use of indicators; according to this aim some indicator type will be involved:

- Quantitative, to assess the impact in intermodal flows;
- General development, to assess the impact on regional economy;
- Conformity and exercise, to assess if the action has been correctly implemented or how the system answered the action;

About each indicator will be expressed:

- Its definition;

- The data sources;
- The covered time range;
- The objective to be satisfied;
- The responsible of monitoring and data research.

Here it is reported a possible series of strategies detailed in this phase. Their practicability is related to coherence with the particular policy evolution at the different territorial levels and with specific structural and functional conditions of the regional transport system:

- Improvement of the infrastructure connections between port, railway and road systems in t territory involved;
- Structural adequacy of the road network and elimination of throttling in the system, especially o the Augusta–Catania–Messina and Trapani–Palermo–Messina railway lines;
- Recovery of the operative possibilities of the ports (stocking spaces, quays for the movement and parking of goods, food stores, etc.);
- Realisation of intermodality in the Augusta and Trapani ports which present adequate approaches, land spaces and possibility of connection with the main traffic routes in the Euro-Mediterranean southern corridor.
- Functional integration in the elements of the Sicilian eastern logistic district and coheren administrative actions for the promotion of the coordination of the Catania, Messina, August and Syracuse ports.

The methodology for the assessment of the transport costs is different from that one for the tim because the observed cost changes according to the considered decision-maker. Generally, therefore, there are costs relative to:

Production of transport service;
Purchase in transport service.

The first case it's referred to peculiar costs of the ship-owner in order to produce the transport, while the second one it's referred to costs that companies producing or purchasing goods support in order carry out the transport from places of origin to those of destination.

Identification of a model to develop the intermodal FREIGHT transportation system in the Euro-Mediterranean Corridor

In this phase ReMoMed project has the purpose to produce an ideal logistic model for developing th intermodal system of goods transportation in the assigned territory for each Region.

In the case of Sicily, the aim of the project will be the definition of a logistic district between Catania and Syracuse for the trade of East Sicily.

Thus a new intermodal layout, understood like a management diagram of the logistic model, will be identified, comparing it with the actual intermodal system. So the necessary actions and strategies should be identified in order to creating the Logistic District Catania-Syracuse.

The purpose of the first action is identification of the logistic hubs and infrastructures that will form the supporting structure of the Logistic District, located between Catania and Syracuse.

The methodology develops the following action in three analysis sections:

1) Overall identification and assessment of the location and interconnections of the aforementioned intermodal logistics hubs, i.e. a global perspective for the Autonomous Community (layout).
2) Detailed identification and description of the various transport and logistics infrastructures composing each of the hubs identified and a description of the route the goods take through th hubs.
3) General assessment of the current situation, of the development requirements identified and of the principal actions to be taken to establish the "ideal" model with respect to intermodal logistics hubs and infrastructure.

The area of the Logistic District concentrates a important part of the population and business in th Region, and it is the trade centre of the East Sicily too.

Besides, the main commercial traffic from the provinces of Ragusa, Enna and Caltanissetta, and from any Italian and foreign Regions, converges on this zone.

In this integrate logistic network the Interport of Catania-Bicocca plays a main role, because it's planned in Southwest area of Catania city, connected with the railway node of Bicocca. In fact its location allows interacting with the other intermodal hubs for the goods transportation, with the haulage terminals, the ports and airports according to the logistic system illustrated in the following diagram.

Therefore the creation of the interport of Catania-Bicocca is decisive for the implementation of the logistic district, in order to optimise all the phases of the productive and distribution process, and to integrate the four transport modalities (railway, road, shipping, air).

Catania Interport will be located near the intermodal terminal of Bicocca, on which the following railways converge:

- Syracuse-Catania-Messina (commercial network);
- Catania-Caltanissetta-Palermo (integrative network);
- Catania-Caltagirone-Gela (local network).

The Interport will be connected with the port by the road, interacting with the national and international shipping, especially in the Mediterranean Sea.

Catania Port is located only 1 Km from the railway station, 500 m from the city centre and 4 Km from the airport. It's connected:

- With the Catania-Palermo Highway A1 8 through the service axis;
- With the Messina-Catania Highway A1 9 through the service axis and the ring road of Catania;
- With the road 1 1 4 Catania-Syracuse;
- With the road 41 7 Catania-Gela.
- Moreover, it's placed near the main Mediterranean ship routes and near the ones connecting the European and Asian ports going through the Canal of Suez. It represents the natural outlet from South Europe to Africa.

The Service Axis should connect the freight village of Bicocca with the main commercial and productive areas. The external road accessibility would insure by the two Highways A19 and A18, as well as the other main roads.

Analogous synergies can exist between Interport and Airport of Catania, where the work in progress for the enlargement of the actual cargo terminal will allow the daily connection with important Italian, European and Mediterranean cities.

About the sea transportation, the development and empowerment of islander economy can promote the short sea shipping in the Mediterranean Sea, especially towards the possible development of the trade with Magreb Countries.

Besides, Catania Port is located between the Tyrrhenian and Adriatic corridors representing the supporting structure of the plan for the economic integration of European Union with South-East Mediterranean Sea.

Catania-Bicocca Interport, on building, is the only one freight village planned in the district area. It's subdivided in two different zones:

- The intermodal centre, or rather the management core of the freight village by level of offered services and use features; inside the main activity is essentially the intermodal exchange between haulage and railway transportation;
- The logistic centre, containing the infrastructures for the logistic and intermodal use, subdivided in external (equipped service areas) and internal surfaces (motor transport stores) with the support service to the persons and to the equipments.

At the moment in the area of the Logistic District we cannot consider an effective intermodal network. It's because there are not logistic hubs for implementing the intermodality in an organic integrated transport system, with a consequent imbalance of the commercial traffics by haulage, and reducing the possible development of the Sea Highways.

In fact, if we exclude the oil traffic, where we especially need to use the shipping, only the 50% of the import/export in the District uses the sea transportation, instead the 40% uses the road, and less than 10% of the trade is interested by railway, the intermodality movements only 1 % of the trade.

The development strategies taken by Port Authority for the empowerment of Catania port are planned to intensify its feature of multifunctional terminal, with a balanced growth of the main activity sectors (commercial, passenger and cruise transport,) and integrating itself with the support of intermodal infrastructures.

Through opportune actions for adjusting the railway network around the interport, this logistic platform will extend its connections with the other transport infrastructures, and it will allow the routing of complete trains directly to the European freeways.

However the commercial function of the Augusta port is actually still unimportant as regards the global traffic of the terminal.

The basic contents of the document containing the results of this phase should include the following sections of analysis:

3.1 Sea Transportation

The development planning of Catania Port is about an adequate and functional design for container traffic and ro-ro.

The strategic development planning consist of:

- An alliance with other near ports, and above all with Pozzallo port, according the guide lines of General Regulator Plan;
- Better integrated logistic services;
- Sea Highways development.

3.2 Railway and intermoal transport

A partnership between "State's railway" and the main actors of railway logistic system (CEMAT Spa. and OMNIA LOGISTIC Spa), have a primary role in Catania-Syracuse district:

- CEMAT Spa., manages road-railway combined transport in 34 terminals in Italy, and in particular, in the four Sicilian terminals of Bicocca, Gela, Milazzo and Palermo-Brancaccio.
- Omnia Logistica Spa, a society of State's railway, works in 1 9 logistic platforms in Italy, so to assure goods transport in 8.000 Italian and European towns. It guarantees own trains, services and a goods network of partners to the clients, about:
  - Centralized department store;
  - Distribution network;
  - Fuel providing for intermodal transport;
  - Informatics systems.

A great weakness of Catania-Syracuse Logistic District is represented by difference between import and export; in fact, in railway transport, import is about 65% and export is about 30%. Other 5% goes inside the Sicily. This is the greatest problem of railway transport in Catania area, and a great obstacle to the development. In fact, in the direction north-south we have trains full at 90%, but in the direction south-north we have trains at 70% of charge.

3.3 Road transport

The absence of an adequate intermodal system promotes haulage and represents the main cause of great number of auto transporters. A research about regional transport system shows that about 88% of tons move on the road, 8% by shipping and 4% by railway. Railway transport is also weak about middle and long distance, and it causes a necessary development of haulage.

3.4 Air transport

In the last period, air transport cargo has shown a great growth, especially because of Etna Valley district and its product (electronic and technologic).

Catania interport is located at the end of one of the most important plurimodal corridor of the PGT ("Tyrrhenian Dorsal"), and it's very interesting for Sicilian goods transport and its trade relationship with middle and northern Italy. In the South of Italy there are three great projects on building: Catania interport

(Sicily), Salerno-Pontecagnano Interport in Campania and Brindisi-Francavilla Fontana interport in Puglia. These three projects represent an important result for logistic system of sea transport, above all in Mediterranean area, and for support to Marcianise-Nola and Gioia Tauro intermodal terminals. The presence of a good link between railway, port and airport represents a possible development of logistic system in Catania and in South-East Sicily. The new Catania interport consists in two great nodes: intermodal centre and logistic centre.

The intermodal centre is about 1 20.000 m^2, it's the place in which move exchange between different transport mode, road-railway and road-road.

The logistic centre is about 1 50.000 m^2, in which there are the most of service to support goods and people.

Analysis of this study shows an actual and future organization of logistic system of Catania-Bicocca and South East Sicily; it considers Catania interport as a valid and necessary intermodal node for South Italy and for euro-Mediterranean corridor.

Catania interport could represent an important intermodal centre also for goods traffic to/from Catania and Augusta ports, and Fontanarossa and Comiso airports. But it's necessary a development also of actual infrastructure, so to implement all the Logistic District of Catania-Syracuse (see Figure 1). In this case it is fundamental the development of functional integration of Augusta port with other infrastructure (road, highway and railway). Catania port, for example, could develop actions with Greece and above all with Patrasso port.

A strategic policy for logistic system represents the born in Augusta port of feeder services so to create a new system "hub and spoke" with Gioia Tauro interport.

It's also possible the development of trade link (ro-ro e lo-lo) between Sicilian ports of Catania and Augusta and other intermodal nodes like Nola, Marcianise and Salerno.

A particularly meaningful indicator of the regional and national role of the logistic District is supplied from the value of its import-export movements with the foreign countries and of that one of its area of gravitation, as regards the more important goods categories. The values of this interchange in 2005 evidence how much important is the economic weight of the Catania-Syracuse District in the Region, to which it refers directly nearly 50% of the value of the imports and nearly 69% of that one of the exports, with 40% of the exports of the agricultural products and beyond 70% of the handcrafted products exported from the Sicily, between which a dominant role have the oil and chemical products.

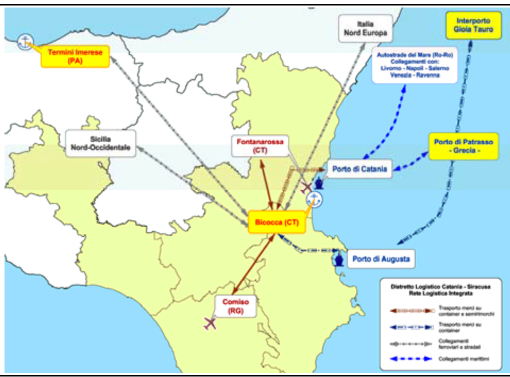


Figure 1 - Relations between intermodal nodes of logistic district CT-SR

The development of the combined and intermodal transport services needs to deepen the identification of the usefulness that allow integrated use of different modes and/or means, and therefore the formation of the new actor like the Multimodal Transport Operator (MTO), that should play a key role for the determination of routes, exchange places, times, transport costs at the national and international scale. The use of many modes and/or means for a same movement implies the transfer, as well as the knowledge of the complementary modes/means and therefore a specific organization that, carrying big flows of traffic, allows economically managing the passages between the modes/means.

In fact the possibility to use high capacity mean on the long distance with reduced costs for distance unit, together with the possibility to concentrate in some terminals impressive amount of standard carg units, using automatic transport units with high power, allowed the big transport operator to organize the shipment flows using many modes and means and reducing the costs. For this reason every goods, in order to carry out a transport, can travel on different modes and/or means in the same way.

Thus, it needs to study in depth, beyond the used cargo, the modes and the means employed. In orde to determinate some indicator which can show the functioning level of the whole system road-railway-shipping, we will adopt the definition "complex monomodal cycle" (for example the oceanic transhipment with the final rotes on feeder vessels) for representing a monomodal transport system b various means, and the definition "multimodal cycle" (for example road-railway-transport) for th intermodal transport by various means.

The current phase of the study puts into context the strategies of development for creating and activating the Logistic District Catania-Syracuse, which model should be shared, experimented and checked in the final phase of the ReMoMed project.

It's still to consider the presence of near important transhipment terminals like Gioia Tauro, Cagliai and Taranto, compared to which a strategy based on local actions isn't useful to the regional and Mediterranean development. Therefore it needs to implement complementary services for the existin hubs, such as Gioia Tauro Terminal, empowering the logistic coordination with the near ports equipped with infrastructures and surfaces having possibility of growth. In fact the absence of cargo breaking in the neighbourhood of Gioia Tauro Terminal and its limited expanding potential suggests developing a feede service through the port of Augusta.

It needs to boost some actions converging the flows in the local intermodal terminal (Bicocca interport) capable of concentrating and distributing the goods on the main traffic corridors, reducing t transport times, optimising the itineraries and then reducing the transport cost. Finally, some priorities of intervention are prefigured about the interior tidy-up rather than the interregional coordination; for reasons, it needs to speed up the improvement process of the infrastructure network too, insuring a adequate service standard of the railways and completing some road axis in Logistic District Catania-Syracuse.

4. Pilot action / model experimentation

The specific objective for the Sicily Region, is been to verify the logistic model presented in the ReMoMed project and to experiment a pilot action in the logistic district of the Eastern Sicily.

The Catania – Syracuse logistic district is one of the most populated and active region of the island: with its road, railway, port and airport system it covers an essential function in transport sector being one of the main gateway of Sicily. It involves Ragusa, Enna e Caltanissetta traffics and Catania and Syracuse ones that arrive from the other Italian and aboard regions. Also important is the role of this district in t commercial and productive field according to the presence of:

- Some areas of intense agro industrial production, in particular of citrus, fruits, vegetables, an wine-producing;
- The main petrochemical refining hub of the Mediterranean, with its barycentre in 30 km o industrial systems, chemical petrochemical and shipbuilding;
- The greater concentration of technology industries of communication and information, by which the "Etna Valley", and a diffuse system of enterprises in many fields, but with particular regard for those of agriculture, upper-middle technology mechanics and informatics;
- Two of the main Sicilian industrial areas (Catania: Pantano D'Arci , Piano Tavola, Tre Fontane; Syracuse: Syracuse – Augusta, Nord-Ovest Priolo);
- Some of the main entrepreneurial Sicilian micro systems;
- Important business activities, with area of interest that involves all the Central Eastern Sicily beyond that the near countries, such as Malta, Tunisia and Libya.

4.1 Economic analysis of enterprises of Catania – Syracuse logistic district

In the time beneath the last two censuses, the Catania Economy registered an increase in the enterprises units of the 25%. In spite of this the employment rate increased the 1,6% only. This increase regards mainly services and hotels, while employment rate decreased in productive field and was stable in building industry.

Sector list for local units and employers:

1° Commerce,

2° Services,

3° Industry,

4° Industry of building.

As regards foreign countries the Catania province, in 2003, exported to them for 865 million, mainly with Europe (55,4%), Asia (34,3%) and America (4%).

As regards import, the exchange volume was of 842 million, mainly whit Europe (59,1 %) and America (19,5%). There are no prevalent goods in the export field.

There are some difficulties in connection with aboard markets so the structure remains related with the inner one.

As regards infrastructure there are some lacks in the road and railway systems, while the port one is sufficient but present a weak arrangement in logistic systems.

The commercial chain

As regards great distribution, in many cases (Germany, France) it has been located in extra-urban or sub-urban areas.

In Italy, mainly in the Southern part, distribution is de-located uniformly in the territory.

As regards the province of Catania the 77% of the commercial activities is located inside the municipalities, so producing relevant flows from the suburban areas to the cities. Some measures have been proposed to solve the problem:

- The creation of another commercial agro industrial node in the Southern part of Catania (near to the railway station and the future interport;
- The destination of a 110 ha area, near the entrance of the Catania city, to detail market and support services activities.

The agro industrial chain is mainly related with the red orange commercialisation. It will be influenced by the adoption of the IGP brand.

The Italian citrus exportation starts mainly from Sicily and it use the auto-transport mode so to congest some road axes of the island.

The Sicilian Capital, Palermo, wasn't able to react to the 90's crisis that involved mainly the commerce and building sectors, while the City of Catania seized the opportunity to develop innovative ones, such as electronic, information technologies, etc..(Ruggiero e Scrofani, 2001). Many local enterprises developed and international ones arrived in the area that was so called the "Etna Valley".

4.2 Data elaboration

Here is presented a survey of the effective transport demand between the Catania-Syracuse logistic district and the Europeans countries partners of the project. This is in the light of the future pilot experiment implementation. First analysis involves the sea transports between the Catania-Syracuse logistic district and the Europeans countries partners of the project, then it considers the whole of the exchanges of the analysed area.

The main product exchanged in Sicily is oil which is moved through the ports of Augusta, Santa Panagia, Gela and Milazzo. As regards the other sectors, in the last years, the electronic components exportation increased while the food and fish one decreased.

This trend involves also the exchange whit the countries partners of the project Re.Mo.Med., such as Spain and Greece, with the participation of the Cartagena Harbour, the Murcia Region and the Western Greece.

As regards Spain the Southern Italy involves the 1 0% of the import and the 1 5% of the export, whi Sicily only the 2,1 % of the total goods exchanged. Another question is the one of oil related products, fo which Sicily is the third exportation region in Italy.

The Spain import in Sicily is mainly constituted by machineries, transport materials and food products and distributed between the Sicilian provinces as follows: Palermo 56,1 %, Catania 1 3,1 % and Messi 8,5%.

Mineral fuels and chemicals mainly constitute the Sicilian export to Spain, while food products cove only the 1 1 ,4% of the regional rate.

It is interesting to see how the 70 % of this export passes through the Catania-Syracuse logistic district. Mineral fuels are movement through the ports of Augusta, Santa Panagia and Milazzo, while chemicals through the provinces of Catania, Ragusa, Syracuse and Messina.

It is to underline how Spain is one of the most important countries of exportation for the Caltanissett and Ragusa provinces.

As regards Greece, the 5% and 3% of the Italian import and export with this country with a value respectively of 209 and 71 mln euro interests Sicily.

According to the data, the main C.T.C.I. categories imported in Sicily by Greece are machineries and transport material (75%, mainly directed to the Palermo province), food products (1 2,4%) and produ by raw materials (7,8%, mainly directed to the Catania province).

The analysis of the Sicilian export to Greece highlights the productive role of Eastern Sicily with the export meanly of mineral fuels and chemicals (78,8% of the total regional rate), distributed between th several provinces as follows: Syracuse (59,3%), Caltanissetta (1 2,1 %), Catania (8,4%) and Messina (8,3%).

- **Excanges bet een te port of Catania and Greece (-)**

During last years in the port of Catania the number of shipments from or direct to Greece increased.

The imported goods are mostly corn, iron, steel products. Instead there are no data regarding to the ships directed to Greece, according to the fact that they called at Catania only for technical reasons.

- **Exc anges bet een t e port of Catania and Spain (-)**

During last years the number of shipments from or direct to Spain in the port of Catania was mor relevant than in the case of Greece, but without the increase revealed in that case. The imported goods are mostly corn, barley, wheat, bananas and paper. There are no data regarding ships directed to Spain.

- **Excanges bet een te port of Augusta and Spain and Greece (-)**

There are no exchanged with Greece while 250000 t per Years starts from Augusta to reach Spain.

. Trade s ech anges between Sicily and Euro-Mediterranean channel Areas

In the analysis of the exchanges of the goods between Sicily and Euro-Mediterranean channel Are some data aggregations were made: countries of the Euro-Mediterranean channel were combined in te macro-areas and goods were joined into two classes: solid (GS) and others (MG, liquids were no considered to highlights the transport demand involved in the pilot experimentation).

Some considerations have to be done:

1 .) Transport mode is the way in which goods entered into or leave the country;

2.) There is a "no declared" item, according to the possibility of the transport operators to not declare the transport mode under some thresholds defined in the CEE 1 901 /2000 regulations;

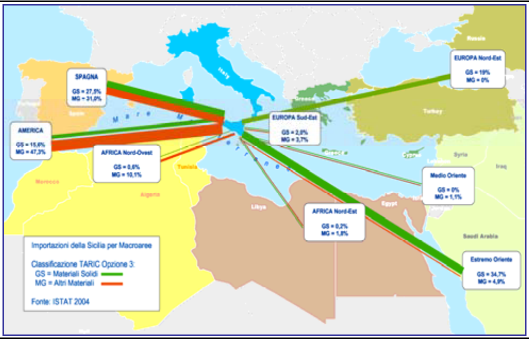
3.) To make data comparable, series includes from 1 991 in the EU1 5 the States annexed in EU on in 1 995 and the Overseas departments annexed in France only in 1 997;

4.) From 1 991 East Central Europe involves: Albania, Byelorussia, Bosnia-Herzegovina, Bulgari Croatia, Estonia, Leetonia, Lithuania, Macedonia, Moldavia, Polonia, Cetz Republic, Romania, Russia, Serbia and Montenegro, Slovacchia, Slovenia, Ukraine and Unghery;

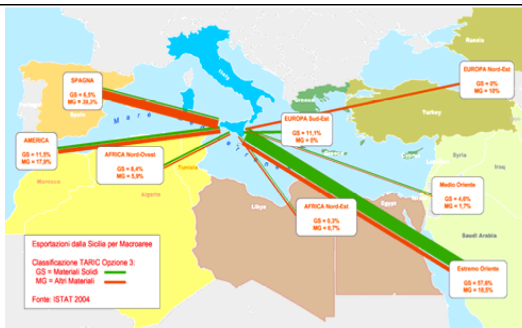
5.) From 1 991 Others Europe countries involves: Andorra, Cyprus, City del Vatican, Gibilter Island, Faeroer Islands, Liechtenstein, Malta, Norway, Switzerland e Turkey.

As regards the Euro-Mediterranean channel, an analysis of the Sicilian exchanges whit eight areas more interested was conduct (see Fig. 2 and 3).

Data were aggregated through the international nomenclature TARIC.



*F*igure 2 - Share value of Sicilian imports from eight areas of the Euro-Med. channel



*F*igure 3 Share value of Sicilian exports to macro areas of the Euro-Med. Channel

4.4 Development objectives and Pilot Project of feeder connections to the Augusta Port

In the previous phase of the project, phase V, some measures were individuated to revamp the integrated infrastructure transport system of the logistic district of Catania-Syracuse. In the present chapter is proposed a pilot project whose simulation is useful to test and assess those measures.

The intent is to adopt a consistent logistic model provided whit efficient infrastructure interconnection.

The study focused on the Augusta port, considering its central role in the Euro-Mediterranean Area.

The pilot project involves a feeder connection between the Hub port of Gioia Tauro and the ports of Augusta, Patrasso and Cartagena (Figure 4).

The pilot project steps are:

- Definition of the Augusta port role in the logistic district of Catania-Syracuse (also considering the potential related to its position);
- Actual scenario assessment through the use of indicators;
- Pilot project scenario assessment through the use of the same indicators;
- Comparing times and costs between the two scenarios;
- Economic assessment.

In the following paragraphs the study results are presented.

*F*igure 4 - Pilot project: feeder connections to the Augusta Port.

4.5 The Augusta Port role

The great potential of the Augusta port is related to its favourable position in the Catania-Syracuse logistic district, to its deep found, to the large squares of earth for service zone, to the interconnection possibility with linear transport infrastructures and, moreover, to the financial aids earmarked for the commercial port.

The project of the Augusta port upgrade is part of a greater regional plan that, under the national and European directives, aims to incentive the road/sea combined transport in order to reduce environment impacts of traffic. The area earmarked to the project is included in the administrative district of the Syracuse Province. It is directly connected to the road system, it is also near the railway station and not so far from the international Airport of Catania Fontanarossa (28 Km).

Actually the Sicilian commercial lines are only: the Genova-Palermo, the Napoli-Palermo, the Genova-Termini Imerese, the Livorno-Palermo and the Ravenna-Catania.

A lot of particular plans involves measures to increase intermodal transport through the coastal navigation, so, according to its deep found that allows the passage of coasting ship of great dimension, the Augusta port has been naturally individuated as the next port, after Palermo, Catania and Trapani, for t Sicilian international agro-industrial export.

At the actual condition the Augusta bay between the municipalities of Melilli, Augusta and Priolo, is one of the most important Mediterranean industrial site as it involves the first refining node of crude oil this area (authorized for 17.600 Kton per year. 70 Km road net and 30 km railway one are available inside the industrial site.

The Augusta territory is divided into three parts:

- The Xifonio Port, between Punta Izzo and Punta Cacarella;
- The Megarese Port, between the Northern and Western coast and the Northern and centr breakwater;
- The Priolo Sinus, between the Southern breakwater and the Magnisi peninsula.

The Augusta port upgrade project was made of three parts: the first two involve the commercial po and has been realized during last years, the last involves the extension of the containers docks and th realization of the services for the Ro-Ro and coasting shipment. The plan is part of a series of measures on the infrastructure national system.

The enlargement of the commercial docks of the Augusta port, which actually movements 31 ,6 million tons of liquids and only 671 thousand tons of solids, will allow new connections with the Adriatic harbours and will make the Augusta port a real node of the Sicilian intermodal net. According to the Regional Planning[1], a Sicilian Logistic Platform is advisable, it involves:

- Catania and Termini Imerese interports;
- Eight auto ports in the island;
- Two integrated port systems;
- In the western part: Palermo, Termini Imerese and Trapani;
- In the eastern part: Catania, Augusta and Pozzallo.

Some measures, that are consistent with the national plans, have been adopted for the four transport nets (road, port, airport, railway).

Figure 5 - The Mediterranean Freeways of the Sea

In the context of the Freeways of the Sea (Figure 5) the European Commission developed the Integration project: a three years project with 10 million amount involving 25 societies of 13 Member States. The project aims to make the sea transport more economic through the implementation of new technologies in shipment and load movement. Its main results are:

- new Ro-Ro and ROPAX ships designed for automated terminals;
- the AGV (Automated Guided Vehicle) machine designed to automate the load/unload operation.

The use of the AGV technology could reduce the waiting times on the dock, which is one of the most important weaknesses of the Freeways of the Sea development. It has a capacity of 500 TEU/h and it could operate both on board and on the dock. Moreover it is possible to connect a series of 10 AGV so moving simultaneously 800 tons of containers.

A terminal provided with the Integration technologies allows both a time and cost saving as it could reach a performance of 35.000 TEU per year per hectare in spite of the current values:

- 20.000 – 25.000 of a good LoLo terminal;
- 10.000 – 15.000 of a conventional LoLo terminal
- less than 10.000 of a lot of the European terminals.

As regards the new RoRo and ROPAX ships, some industries (Fincantieri, Izar and Catena) have designed a series of them with a great range of capacity (80-1400 TEU equivalent) and speed (10-55 nodes). The ships were devised both for automated terminal with AGV than for terminal of high performance. This was in the light of secure needs both of the short sea shipping (including feeder shipment) and of inter-modal transport. The Integration technology is more economic and involves the use of a reduced number of ships than in the feeder case. In fact, with the AGV, it is possible to realize a load/unload cycle of a RoRo ship Integration with a 1400 TEU capacity in 6 hours only. Operating in this condition, a terminal could movement 2 million TEU per year involving 14 ships only in spite of the 25 required by the traditional terminal for the same volume of movement.

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[1] "Executive Plan of freights transport and of the logistic" approved by the Regional decree n. 33/04 (G.U.R.S. n. 11 of 12-03-2004).

4.6 Experimenta l impl emen tation the development proposed model

The development proposed model considers both the previous survey and the actual condition o Sicilian Ro-Ro and Container feeder lines.

It is useful firstly to analyse the current condition of connections between Southern Italy and Spain and Greece. The current condition is characterized by high travelling times and low efficiency. As example it is reported the Palermo-Piraeus line timetable (see the figure below) with 1 2 calls for a 25 days travelling time (table 1).

Moreover the only direct connections are between Palermo and the Ports of Valencia and Barcelon In the table 2 is represents the valuation of the cost of these connection based on the used methodology.

The valuated elementary costs are:

- 0,81 1 €/sea mile for Ro-Ro shipments of 1 8,00 m lorry;
- 0,362 €/sea mile for shipments of 20’ container;
- 0,676 €/sea mile for shipments of 40’ container.

Table 1

Ports of call	Cumulated days
Palermo (I)	-
Salerno (I)	[illegible]
Savona (I)	[illegible]
Setubal (P)	[illegible]
Bristol (GB)	[illegible]
Cork (IRL)	[illegible]
Esbjerg (D[illegible])	[illegible]
[illegible]allhamn (S)	3[illegible]
Antwerp (B)	[illegible]
Southampton (GB)	[illegible]
Salerno (I)	[illegible]
Malta (ML)	4[illegible]
Pireo (GR)	[illegible]

Table 2

Path row	Distance (sea miles)	Frequency	Estimated price (euro)	Estimated price Feder service (euro)	
			RO-RO [illegible] m	Container [illegible]’	Container 4[illegible]’
Palermo -Valencia	[illegible]	[illegible]x week	3[illegible]	3[illegible]	43[illegible]
Palermo-Barcelona	[illegible]	[illegible]x week	-	4[illegible]	4[illegible]

The study focused on the Augusta port, considering its central role in the Euro-Mediterranean Area. The pilot project involves a feeder connection between the Hub of Gioia Tauro and the ports of Augusta Patrasso and Cartagena.

The pilot project steps are:

Definition of the Augusta port role in the logistic district of Catania-Syracuse (also considering th potential related to its position);

Assessment of the actual scenario through the use of indicators;

Assessment of the pilot project scenario through the use of the same indicators;

Comparing times and costs between the two scenarios;

Economic assessment.

Actually there are no direct possibilities of export from the logistic district, so goods are transferred to the Palermo Port with further road movement costs and stoking problems according to the inadequacy of the facilities of this harbour. This condition causes also environmental impacts and insufficient security in the Sicilian road network.

The project for upgrading the Augusta port is part of a greater regional plan that, under the natio and European directives, aims to incentive the road/sea combined transport in order to reduce environmental impacts of traffic. The Augusta port upgrading project was made by three parts: the first two steps involve the commercial port and have been realized during last years, the last part involves the extension of the containers docks and the realization of the services for the Ro-Ro and coasting shipment. The plan is part of a series of measures on the infrastructure national system. The enlargement of the commercial quay of the Augusta port, which actually movements 31 ,6 million tons of liquids and only 671 thousand tons of solids, will allow new connections with the Adriatic corridor and will make the Augusta port a real node of the Sicilian intermodal system.

In an optimistic perspective, with the actually available docks and spaces, two ships per day with 500 TEU could land to the Ro-Ro terminal. The terminal should work 24 hours per day and movement 700.000 TEU equivalent per year. The Augusta port is ready to be involved in the Freeways of the Sea Net and to be completed with the Ro-Ro terminal. The development model proposed considers both the previous survey and the actual condition of Sicilian Ro-Ro and Container feeder lines connecting Southern Italy to Spain and Greece. The current condition is characterized by high travelling times and low efficiency. Moreover the only direct connections are between Palermo and the Ports of Valencia and Barcelona.

The valuation of the costs of the Ro-Ro shipments and Container feeder ones confirms an advantage in the feeder shipment in spite of the classic Ro-Ro one, but it has to be taken into account that the implementation of new Ro-Ro technologies (Integration AGV) could reverse this condition.

The actual perspectives of the Augusta Port are related to its upgrading project. The third part of the project, that involves the extension of the container docks and the realization of the services for the Ro-Ro and coasting shipment, previews a 240.000 container/year movement. Considering that only a share of 20% of these containers is addressed to the TRANSHIP traffic, the rest, 1 92.000 Container/year, will be addressed to the traffic of articulated lorry on the Sicilians road networks.

5. Conclusions

The pilot project aim is to overcome the difficulties expressed before especially as regards the import/export issue in the CT-SR logistic district.

Actually there are no direct possibilities of export from the logistic district, so goods are turned aside the Palermo Port with road movement costs and costing problems according to the inadequacy of the squares of this harbour. This condition causes also environmental impacts and insufficient security in the Sicilian road network.

The pilot project involves a feeder connection between the Hub port of Gioia Tauro and the ports of Augusta, Patrasso and Cartagena. The intent is to give to the Augusta port a relevant role both in the Euro Mediterranean Channel than in the Continental connection of Gioia Tauro. Not at last it was considered the possibility that the Port could help the productive context of the CT-SR district.

In the table 3 and 4 the evaluation of the costs of the Ro-Ro shipments and Container feeder ones with the methodology before showed are reported (hypothetic routes were considered to assess the distances between the ports).

Table 3

Path row	*Distance □sea miles□*	*Estimated price of RO-RO □1m ser vice □euro□*
Augusta – Patrasso	*31□*	*4□*
Augusta - Gioia Tauro	*4□*	*33□*
Augusta - Cartagena	*4□*	*3□*

Valued costs confirm the connections economic advantages. Moreover there are to be considered the environmental and security advantages of reducing the traffic in the SR-CT-PA line.

Table 4

Path row	*Distance □sea miles□*	*Estimated price of Feeder service □euro□*	
		Container □□'	*Container 4□'*
Augusta - Patrasso	*31□*	*1□*	*3□*
Augusta - Gioia Tauro	*4□*	*4□*	*1□*
Augusta - Cartagena	*4□*	*3□*	*4□*

The valuated elementary costs for the Augusta Patrasso line are:

- 1 ,421 €/sea mile for Ro-Ro shipments of 1 8,00 m lorry;
- 0,522 €/sea mile for shipments of 20' container;
- 1 ,006 €/sea mile for shipments of 40' container.

These values highlight an advantage in the feedershipment in spite of the classic Ro-Ro one, but it has to be taken into account that the implementation of new Ro-Ro technologies (Integration AGV) could reverse this condition.

The actual perspectives of the Augusta Port are related to its upgrade project. The third part of the project, that involves the extension of the container docks and the realization of the services for the Ro-Ro and coasting shipment, previews a 240.000 container/year movement. Considering that only a share o 20% of these containers is addressed to the TRANSHIP traffic, the rest, 1 92.000 Container/year, will be addressed to the articulated lorry traffic on the Sicilians nets.

In the previous phase of the project some measures were individuated to revamp the integrated infrastructure transport system of the logistic district of Catania-Syracuse. In the present chapter is proposed a pilot project whose simulation is useful to test and assess those measures. The pilot project involves a feeder connection between the Hub port of Gioia Tauro and the ports of Augusta, Patrasso and Cartagena(Fig.4).

The pilot project steps are:

- Definition of the Augusta port role in the logistic district of Catania-Syracuse (also considerin the potential related to its position);
- Actual scenario assessment through the use of indicators;
- Pilot project scenario assessment through the use of the same indicators;
- Comparing times and costs between the two scenarios;
- Economic assessment.

Then the study results were presented. Also the economic advantages of the connection proposed were investigated, having in mind that the general context of the connections in the Euro Mediterranean Are examined, is not of easy simulation and needs a deep knowledge on the economic- entrepreneurial conditions of all the stakeholders (Shipment companies, Transport enterprises, Management societies o the ports).

According to a number of infrastructures, logistic, territorial reasons expressed in the present study, the best localization of this node coincides with the Augusta Port.

The general context of the connections in the Euro Mediterranean Areas examined is not easy to model and needs a deep knowledge on the economic and entrepreneurial conditions of all the stakeholders (Shipment companies, Transprt enterprises, Management societies of the ports). Overall the need of an intermodal node is expressed in the CT-SR logistic district, especially of a port terminal that is able to interface itself with the others of the Mediterranean port systems.

According to a number of infrastructure, logistic, territorial reasons expressed in the present study, the best localization of this node coincides with the Augusta Port.

The study focused on the Augusta port, considering its central role in the Euro-Mediterranean Area. The pilot project involves a feeder connection between the Hub port of Gioia Tauro and the ports o Augusta, Patrasso and Cartagena.

The pilot project steps are:

- Definition of the Augusta port role in the logistic district of Catania-Syracuse;
- Actual scenario assessment through the use of indicators;
- Pilot project scenario assessment through the use of the same indicators;
- Comparing times and costs between the two scenarios;
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A MARKOVIAN MODEL AS A TOOL FOR OPTIMIZING RAILWAY TRACK MAINTENANCE

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1. Abstract

This paper is designed to develop a procedure which defines a planning criterion for railway superstructure maintenance by means of Markov decision processes. This methodology allows to formulate a specific policy π which carries out the best configuration of budget allocation (min. Φ), and at the same time to guarantee the highest efficiency level in the railway superstructure.
Thanks to the dynamic programming technique applied to decision processes, the report has examined the possibility of establishing the best management policy in order to maintain adequate safety levels of implementation and quality speed levels, in the presence of budget constraints, thus optimizing the available resources.
This procedure therefore allows to optimize the maintenance protocols currently adopted which ensure the superstructure efficiency state through an *on condition* maintenance, based on the surveys of the track geometric parameters conducted on high-performance diagnostic carriages.
Finally, a case study has been developed in order to estabilish the application limits of the methodology suggested in order to formulate the best strategy for managing the resources and improving speed quality and safety.

2. Foreword

In optimizing road or railway maintenance, it is necessary to duly analyse on the one hand the relations between the sets of variables which determine the safety conditions of the superstructure, and on the other hand, to implement suitable intervention strategies which optimise the resource allocation in order to ensure the highest functional efficiency level of the infrastructure.
From the methodological perspective it is, therefore, possible to tackle the problem systematically by means of stochastic dynamic programming, and especially Markov decision processes. As a matter of fact, these models allow to assume a finite number of probable states according to which the infrastructural system is classified, and to refer to each i-th alternative of intervention as ‘optimal’ in terms of efficiency and cost effectiveness by means

of a monitoring procedure suitable to detect the degree of deterioration of the system at any time.

The conditions of deterioration of a superstructure are known to be expressed in terms of the values assumed by such geometric and wear parameters as the *G*auge, the Alignment, the Longitudinal Level, the Transversal Level, the Transversal Level **D**eviation, the Superelevation **D**efect, the Skew, and the Wear of *V*ertical, Horizontal and 45 ° **R**ail which, as a function of the time, determine the state of a superstructure in terms of efficiency or degree of deterioration.

Moreover, by analogy with other fields of the transport engineering, some synthetic indexes can be used in order to summarize the operational state of the line with regard to the geometric configuration at the time of the survey. The *track quality indexes* (**IQ**) allow to assess "the effectiveness of the line" by obtaining a single coefficient from a suitable elaboration of the above-mentioned geometric parameters.

Such a mathematical approach allows to carry out, in the optimization study, the aggregate analysis of the sets of variables which altogether characterise the quality of the service for the user (in terms of comfort and safety), as well as to provide, also from a mathematical perspective, the solutions of the decision making process and therefore, the optimal intervention policies with regard to the probabilities of deterioration of the state variables of the infrastructure over time, to the flow of the available resources and to the global efficiency of the interventions.

3 The geometric and wear parameters of the rail superstructure

The dynamic loads exerted by the carriages during the operation (vertical or lateral actions, etc.), the thermal loads and the physiological deterioration of the materials, by interacting synergically between them, cause the deterioration of a rail superstructure meant as loss of the planned geometric configuration (in that case called geometric defect of track 1), wear of the single track component and of the superstructure as a whole.

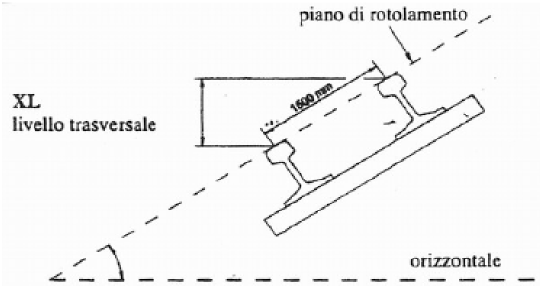
As previously stated, the track *geometric parameters* which have to be taken under strict control in that they have a direct influence on the rail traffic safety, are the following:

G*auge* The gauge is defined as the distance, expressed in mm, between the inner faces of the rails of the two railway tracks, measured 14 mm below the rolling surface.

Alignment "A" The alignment is the measurement, expressed in mm on the plane parallel to the rolling plane, of the distance between the inner face of the rail and a line of a given length, which joins two other points equidistant from the point to be measured; the distance between the two reference points (line length) is equal to 10 m. In technical jargon, the alignment is also termed as "arrow".

Longitudinal Level “L” The longitudinal level is the measurement, expressed in mm on the longitudinal vertical plane, of the distance between the rolling board and a line of a given length which joins two other points equidistant from the point to be measured; the distance between the two points (line length) is equal to 10 m.

Transversal Level “XL” The transversal level, expressed in mm, is obtained as a value of difference in height between two adjacent rolling boards; it coincides with the height of a right-angled triangle whose hypotenuse is equal to 1500 mm and a vertical angle is equal to the angle between the rolling plane and a horizontal reference plane.



Transversal level deviation “SCARTXL” At a certain point, the transversal level deviation is defined as the value of the difference, expressed in mm, between the point transversal level and the mean of the transversal levels XL of two points, respectively set 5 m before and 5 m after the considered point;

Supraelevation Defect “ΔH” The supraelevation defect in a given point of the track axis is the measurement, expressed in millimetres, of the difference between the transversal level XL and the planned supraelevation h. The latter parameter is equal to 0 in straight-line sections or on curves of wide radius (in the absence of supraelevation) and respective planimetric transition joints, while it assumes a nonzero value in the curvilinear sections (constant h) and in the respective joints (variable h).

Skew “γ” The skew γ is defined as the inclination, expressed ‘by 1000’, of a rail with respect to the other, calculated as a ratio of the difference of transversal level XL between two track sections set at a given distance, termed as skew gauge length, to the gauge length itself. This gauge length can be equal to 9 m or 2,5.

Track wear The wear can be defined as the lack of material, expressed in mm, with respect to the theoretical profile of the track itself. It is vertical if relative to the rolling surface of the track and measured along the symmetry axis of the transversal section; horizontal if relative to the inner face of the rail and measured on the straight line intersecting “the vertexes of the joint planes of the rail”; ant at 45° if measured on the straight line intersecting the joint radius drawn between the face and the upper surface of the rail and inclined at an angle of 45° with respect to the axis of the track.

There are still additional parameters which cannot directly be measured but deduced from those previously mentioned, "manually" or by means of software: the Moving average gauge "S_{100}" and the Wear purified gauge.
For high-speed lines (V > 250 Km/h), the following parameters can be employed as control indexes: the Alignment on a 20m b asis "A_{20}", the Longitudinal Level on a 2 0 m basis "L_{20}" and the Transversal Level **D**eviation on a 20m bas is "SCA**R**TXL_{20}".
In addition to the above-mentioned parameters, the operational safety calls for inspection of the state of the tracks in order to detect any hole or crevice in the spans, as well as of the thermit and flash weldings.

4. *The maintenance planning of the railw ay superstructure*

By means of the instrumental measures carried out through diagnostic carriages or trains, it is possible to make an assessment of the state of a superstructure by comparing directly the threshold values which differentiate the warning level according to every geometrical parameter under consideration.
The regulations on the survey of the track geometry, made by RFI, provide for three ***track quality levels*** according to which the 'full implementation of the line" is expected and a level which requires some operational restrictions on the railway in the form of slowing-down in the line and traffic block. More specifically, it is stated:

- ***1° quality level:*** in this field the geometry is to be considered in <u>excellent conditions</u>;
- ***2° quality level:*** in this field the geometry is <u>to be monitored</u>: it is indeed required to identify and assess the reasons why the deterioration has occurred, how the defect will evolve over time and finally, to plan maintenance works according to such a rate;
- ***3° quality level*** (or segment of intervention): in this field <u>the implementation of the maintenance works is required within set times</u> or on the basis of the value assumed by the Track Geometry Quality Index;
- ***level involving operational restrictions***: in this case some <u>traffic slowing down or block</u> is required.

The defects which are included in the last level as well as those in the 3rd level calling for immediate intervention (within 48 hours) are defined as relevant defects of the track geometry. These defects are highly dangerous to the railway traffic and therefore must be repaired in the shortest possible time. Should it prove impossible, operational restrictions must be imposed proportionate to the type of the parameter and to the value assumed by the parameter itself. Such restrictions may be traffic slowing down or block.
RFI has identified global service indexes which express the operational level of a line with regard to the geometrical configuration, at a section or a station, by duly processing some of the above-mentioned geometric parameters; notably,

RFI has chosen longitudinal level, alignment and transversal level. As regards skew, gauge and wear at 45°, they are assessed one by one in that they are of vital importance to the safety conditions of a line.
Such track quality indexes (TQI) then allow to assess the "effectiveness of the line", and its evolution over time, so as to plan adequate "on condition" maintenance works, optimizing human and economical resources and avoiding proceeding with a more expensive "logic of 24-hour repair service" to be performed urgently whenever the relevant defects arise.
In order to calculate the quality indexes, RFI regulations on track maintenance specify the following defectiveness indexes:

- defectiveness index of the longitunal level, equal to the standard deviation on a 200m plane of the longitudinal level;
- defectiveness index of the alignment, equal to the standard deviation on a 200m plane of the alignment;
- defectiveness index of the transversal level, equal to the standard deviation on a 200m plane of the transversal level;
- wedging index, equal to the highest on a 200m plane, and therefore to the worst of the above-mentioned defectiveness indexes.

As regards the track quality indexes (IQB), they are defined as follows:

❖ **Track geometry quality index in a section IQBT** is the mean value of the wedging indexes (one at every 200 m) concerning the section considered. It is assessed at a full line, so the tracks at stations and the other places comprising points are excluded.

❖ **Track geometry quality index in a station IQBS** is the mean value of the defectiveness indexes of the longitudinal level (one at every 200 m) at the place considered.

In accordance with the tests conducted by RFI and FICHE UIC 518, the index values are to be below set values depending on the highest speed of the line section considered **Table [illegible]** sums up the values permitted.

INDICI DI QUALITÀ DEL BINARIO		
Intervallo di velocità tratta [Km/h]	Indice di qualità in tratta IQBT [mm]	Indice di qualità stazioni IQBS [mm]
V< 80	3.1	3.7
100<V<100	2.8	3.4
100<V<120	2.6	3.2
120<V<140	2.4	3.0
140<V<160	2.0	2.6
160<V<180	1.9	2.5
180<V<200	1.8	2.4
200<V<250	1.4	20

Table [illegible] Track geometry quality index in a section and in a station

The sections/places showing an excess of the values permitted for the Qualit y Indexes of Section/Station ***(unconformable sections/places***), require programming maintenance works which have to be carried out within the times set in the following *table*

	Tempi di intervento per la manutenzione on condition	
	Supero dell'Indice di Qualità	*Tempo per l'esecuzione*
Linee di Gruppo A	0÷20%	4 mesi
	oltre il 20%	2 mesi
Linee di Gruppo B	0÷20%	8 mesi
	oltre il 20%	4 mesi
Linne dei Gruppi C e D	0÷20%	12 mesi
	oltre il 20%	6 mesi

Table Interv ention times for 'on condition' maintenance

Let the state of the system X_t be defined by the parameters assessed on such geometric and wear characteristics of the railway superstructure as the *G*auge, the Alignment, the Longitudinal Level, the Transversal Level, the Skew, and the Wear of a 45° **R**ail. For each of them the regulations specify the following levels and thresholds:

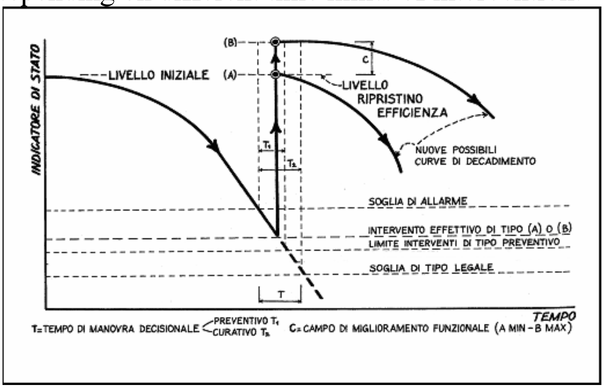
- 1st Quality Level (optimal)
- 2nd Quality Level (warning)
- 3rd Quality Level (short-term or medium-term intervention)
- 4th Quality Level (operational restrictions: slowing down or block)

Each state of the system previously defined can be associated to the undertaken action A_t. The action can be an operational restriction, the implementation of maintenance works or can consist in keeping the superstructure without any intervention up to the next inspection period. In any case, if X_t=i and A_t=k, P_{ij} (k) denotes the probability for the system to be in the state j, at the time t+1, by undertaking action k. Moreover, for each state i and the subsequent action k there will be an estimated cost C(i, k).
The characteristics of the superstructure contribute, as a whole, to the definition of the decay function of each parameter*:*

$$F = (\mathrm{P}, \mathrm{U}, \mathbf{R})$$

As a matter of fact, since railway headquarters also play a leading role, it is clear that every single parameter must be within the limitations provided for by the regulations.
The functional form F will express the decay speed of the structural and functional track characteristics up to the achievement of a minimum threshold F_{min} which corresponds to loss of functionality of the superstructure.
Any maintenance strategy involves, therefore, the preventive knowledge of the state X_t=i of the system, which is known to be defined by the values assumed by the diverse geometric and wear parameters observed.

The high-performance means of measurements allow nowadays to determine the value of such parameters and finally through data post-processing, to define the value of a synthetic indicator of the state of the system under analysis IQB. The IQB levels define however the operational timing of "on condition" maintenance as they are situations which require maintenance works all the same, even if depending on different time limits of intervention.



5. Theoretical observations on markovian stochastic decision processes

A Markov chain is a stochastic process at such finite states (with values in X) $\{X_t\}$(t=1,2...) that, being X_t and A_t the state of the system and the subsequent action in the time t respectively, if $X_t = i$ and $A_t = a$, then denoting with $P_{ij}(a)$ the probability that the system, at the state i at first, will be, in the time $t+1$, at the state j after action a has been undertaken, it follows that:

$$P\{X_{t+1} = j \mid H_t, X_t = i, A_t = a\} = P\{X_{t+1} = j \mid X_t = i, A_t = a\} = P_{ij}(a).$$

This expresses Markov chain's property assuming that the conditional probability $P_{ij}(a)$ of any future 'event', given each past 'event' and the present state $X_t = i$, is independent from the past "event" and from the time sequence H_t of states and actions of the system $H_t = (X_0, A_0, X_1, A_1, \ldots, X_{t-1}, A_{t-1})$ whereas it is only dependent on the present state of the process.

For each $X_t = i$ and $A_t = a$, there will be also an estimated cost $C(i, a)$ which is assumed to be limited (i.e. we decide to operate in the presence of budgetary constraints), then there will be a number M such that:

$$C(i, a) \le M \qquad per\ ogni\ i, a \in$$

The conditions of deterioration of the superstructure are known to be expressed by means of the values assumed by such strategic variables as load-carrying capacity, regularity and wrinkling which, in function of the time, define the state of the superstructure in terms of efficiency or degree of deterioration.

The determination of the state of the system will be followed by an action (decision) selected from a finite set of possible actions. A rule applied for the

choice of an action at each instant is defined Policy and is denoted by π. In general, the action specified by a policy in the time t, may be dependent not only on the present state X_t, but also on the history H_t and on t itself. A policy can be also randomized in the sense that a probability distribution is assigned to possible actions.
An important subset of the set containing all the policies is that of the stationary policies. A policy (or scenario of coordinate activities) is stationary if the action which specifies in the time t is only dependent on the present state X_t and if it is non-randomized (deterministic). In other words, a stationary policy is a function f which traces the state space in the action space by expressing that for each state i, f(i) denotes the action chosen by the policy when the state i occurs. It follows that, if a stationary policy is adopted, then the state sequence $\{X_n, n = 0, 1, 2, \dots\}$ forms a Markov chain with a transition probability $P_{ij} = P_{ij}(f(i))$. Therefore the process will be referred to as Markov decision process.
In order to determine the most effective actions of intervention, it is necessary to establish a criterion for optimality.
Considering the results of the methodology employed in memory, the maintenance policy thought as optimum is that policy which minimizes the average cost per time unit. Such a solution can be obtained by solving a problem of linear programming through the Simplex Method, in which the objective function to minimize is as follows:

$$\Phi = \sum_{j=0}^{m} \sum_{k=1}^{K} x_{jk} c_{jk} \qquad (1)$$

Where, x_{jk} stands for the stationary probability to take a decision k when the system is at the state j, and $c_{jk}(t) \geq 0$ the average cost covered when the system is at the state j in the time t and the decision k is taken. It is assumed that $c_{ik}(t)=c_{ik}$, independent of t.

$$\sum_{k=1}^{K} x_{jk} = \sum_{i \in I} \sum_{k=1}^{K} x_{jk} q_{jk}(k) \qquad \text{for each } j \in I \qquad (2a)$$

$$\sum_{j \in I} \sum_{k=1}^{K} x_{jk} = 1 \qquad (2b)$$

$$x_{jk} \geq 0 \qquad (2c)$$

in which $q_{ij}(k)$ espresses the transition probability of a system, being in the state i, to change to the state j in the following step when action k is taken. For each type of action k the sum of all the probabilities that the system, being in the state i, has to change to each state j = 0,1…m:

$$\sum_{j=0}^{m} q_{ij}(k) = 1 \qquad \text{and} \qquad qij(k) \geq 0 \qquad \text{For each i and k set}$$

As previously shown [21], the optimum solution of (2) is unique for $\sum_k x_{jk} > 0$, j∈I, and it satisfies the optimum non-randomized choice to take action *k* in the state *j*, then it follows:

$$D_{jk} = \frac{x_{jk}}{\pi_j} = \frac{x_{jk}}{\sum_{k=1}^{□} x_{jk}};$$

in which there is, $x_{jk} = \pi_j D_{jk}$ and $\pi_j = \sum_k x_{jk}$, with π_j as equilibrium probability distribution.

□ *Example of an application of markovian decision process to maintenance problems of railway superstructures*

In order to fully describe the real possibility to use the mathematical model suggested and closely follow the application steps of the method, the case of a system characterized by possible states for each geometrical parameter above mentioned, by possible actions to be undertaken and by actions consistent with every state and every action, has been reported as an exemplification.
Considering the state indicators suggested by RFI, let us examine, for the sake of simplicity, the following four states so as to classify the qualitative characteristics of the infrastructure:

State	Deterioration threshold
□	Optimal Level
□	Warning Threshold
□	Intervention Threshold
3	Safety Threshold

Table 3 □ Costs and transition probability for each state and action

where:

- *optimal level*: characteristic value equal to that of a new or efficiently restored superstructure;
- *warning threshold*: characteristic value such as to induce monitoring of the section in order to assess the defect evolution over time;
- *intervention threshold*: conditions of deterioration of the superstructure such as to require implementation of maintenance works within times set in advance;
- *safety threshold*: characteristic values which definitely compromise the safety to such an extent that they require the implementation of traffic slowing down or block.

Let us suppose that at each initial period *t* the infrastructure is inspected by means of high-performance diagnostic carriages and classified according to one

of the four states above considered. After the observation of the qualitative state of the superstructure a decision needs to be taken, either to still utilize it without any intervention up to the following inspection period (action c) or to carry out maintenance works in order to restore the best structural and functional conditions (action m). Let us suppose, moreover, that action c is allowed only at the state 0 while action m is permitted only at the state 3.
For each parameter, from every state to the others, the transition probability may be calculated by knowing its decay function *F* and by carrying out a statistical analysis of the data obtained from the measurements of the structural and functional characteristics (monitoring and survey of the quality indicators).
Let us also assume that choosing action c implies running costs equal to 100, 200 and 500 monetary units for each period, at the states 0, 1 and 2 respectively. Moreover the infrastructure will be at the state j at the beginning of the following period with probability $P_{ij}(c)$, i=0,1,2.

State *i*	*Action* *k*	*Costs* $C(i,k)$	*Transition probability* $P_{i0}(k)$	$P_{i1}(k)$	$P_{i2}(k)$	$P_{i3}(k)$
0	c	100	0.75	0.19	0.06	0
1	c	200	0	0.75	0.19	0.06
	m	2100	0.75	0.19	0.06	0
2	c	500	0	0	0.75	0.25
	m	2600	0.75	0.19	0.06	0
3	m	3100	0.75	0.19	0.06	0

Table 4 – Costs and transition prability for each state and action

Whereas, choosing action m means performing maintenance works in order to restore the optimum level of the superstructure (state 0); maintenance works will involve costs equal to 2100, 2600 and 3100 monetary units depending on which state - 1, 2, 3, respectively - the decision is taken.
Finally, let us assume that the road section is in the state j at the beginning of the next period with probability $P_{ij}(m) = P_{0j}(k)$, i=1,2,3.
The costs C(i,k) and the probabilities $P_{ij}(k)$ are shown in *Table 4*. The aim is to formulate an optimum replacement policy which minimizes the average cost of every period.
Let us assume that k = 0,1 for actions "c" and "m" respectively, and that i=0,1,2,3 the states of the system.
The objective function (4) and the constraints (5) with reference to the problem under consideration will be:

$$\min \Phi = 100x_{00}+200x_{10}+2100x_{11}+500x_{20}+2600x_{21}+3100x_{31}$$

Referring to the initial state j=0 as an example, the condition (5a) may be formally expressed as follows:

$$x_{00}+x_{01} = x_{00}\,P_{00}(0) + x_{01}\,P_{00}(1) +$$
$$x_{10}\,P_{10}(0) + x_{11}\,P_{10}(1) +$$

$x_{20}\ P_{20}(0) + x_{21}\ P_{20}(1) +$
$x_{30}\ P_{30}(0) + x_{31}\ P_{30}(1)$

Thereby, referring to the above-mentioned numerical example, the conditions (5) will be:

$$\sum_i \sum_k x_{ik} = 1$$

$-0.25x_{00}-x_{01}+0.75x_{11}+0.75x_{21}+0.75x_{31} = 0$
$0.1875x_{00}-0.25x_{11}-0.19x_{11}+0.19x_{21}+0.19x_{31} = 0$
$0.06x_{00}+0.19x_{10}+0.06x_{11}-0.25x_{20}-0.94x_{21}+0.06x_{31} = 0$
$0.06x_{10}+0.25x_{20}-x_{30}-x_{31} = 0$
$x_{1k} > 0$ for each i, k

The solution of the problem, obtained by means of the simplex algorithm, will provide the vector **x**□=(x_{ik}) of the limit probability distribution which minimizes the monthly average cost. The vector **x**□ will have the property (Dantzig and Wolfe) according to which for each i, the x_{ik} will be equal to 0 except for a unique value k.
Thus the policy defined by the conditional probability

$$\mathbf{D}^*_{jk} = \frac{x^*_{jk}}{\sum_{k=1}^{K} x^*_{jk}}$$

to take action k in the state j, is non-randomised, that it, the action called for, when being in the state j, is a deterministic function of j.
The solution of the linear programming problem provides the following results:

□***i,k***□	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
x□$_{ik}$	0.429	0.429	0	0	0.116	0.027
D□$_{ik}$	1	1	0	0	1	1

Therefore, the conditional probabilities $\mathbf{D}_{jk}$ derived from the optimum solution define the following Policy **R**: keep the superstructure at the states 0 and 1, carry out maintenance works at the states 2 and 3. The average cost of each period will be equal to 513.39 monetary units.

□ Conclusions

The fundamental objective of a railway maintenance plan is to guarantee, by means of suitable mathematical models for optimizing the effectiveness of the interventions, a support system for the decisions which are required to perform upgrading actions so as to ensure the necessary degree of reliability of the structure and at the same time to optimize the resource allocation.

The study in this report has allowed to provide an appropriate mathematical model which, starting from the integrated analysis of the several components which characterize the degree of degradation of the infrastructure, develops the most suitable intervention policies with regard to speed quality for the user and operational safety at the minimum average cost per time unit (maximum efficiency) and thereby in optimal conditions with regard to the economical resource allocation.
More specifically, dealing with the railway superstructure, a great emphasis has been placed on a complete characterization of the performances assessed by each geometrical parameter of the track and at the third quality level (or intervention), on the calculation of the values assumed by IQB index for programming maintenance works.
The mathematical approach to the problem here proposed refers to Markov's theory of the decision processes and allows a valid analytical interpretation of the decision-making regulations.
The need for a careful analysis of the many technical, economical and environmental aspects in the management stage leading to the optimization of maintenance works, requires to establish appropriate mathematical criteria for a preliminary assessment of the best intervention strategies with regard to the peculiarities of the problem under consideration.
In order to make the results obtained by analysis easy and ready for use, the solution to the linear programming problem has been exemplified with a case study providing the relative results and therefore the characterization of the specific functional relationships which determine the optimum intervention policy by maximizing the objective function in non-randomised conditions (the action is in deterministic function of the state j of the superstructure).

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SAFETY PROBLEMS AND GLOBAL COMFORT IN RAILWAYS OPERATIONAL MANAGEMENT.

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Abstract

In the planning process of a railway intervention and in its operational management, it needs to analyse carefully the relations between the different classes of variables constituting the complex system "vehicle-infrastructure-environment" to find the most suitable actions in optimisation the travelling comfort and safety, with respect for the ecosystem compatibilities.

The problem can be tackled, in a systematic way, according to set theory and vector analysis, in a hyperspace $\Re^d$ of d-dimension up to determining a characteristic *global comfort vector* $\vec{C}^i_g$ that can be combined with each alternatives of intervention in the transport network examined.

From this point of view, after the proper achievement of the other strategic design objectives has been tested (e.g. financial budget, minimization of the generalized costs of transport), it is suggested to consider the particular solution h by which the modulus of the vector $\vec{\mathbf{C}}^i_g$ satisfies the relation $C^h{}_g = \max \{C^i{}_g\}$. The mathematical approach suggested allows to carry out the aggregate analysis of the classes of variables which describe the railway service quality on the whole to define geometrically the relations which exist between the abovementioned indicators through an appropriate transformation of $\Re_d$.

The proposed mathematical model can be extended for the total optimisation of the infrastructural interventions on a railway network, so the information system assumes, according to this point of view, the functionalities of a SDSS (Spatial Decision System Support), for the management of a "models base" and a "data base" for the management and processing of safety, exercise and management data.

1. Study of the interactions between a human being and the vehicle-infrastructure-environment system

In designing an intervention in the transport network (new constructions, upgrading actions, improvement of functionality of the junctions, etc.) in order to guarantee the whole transport system the highest

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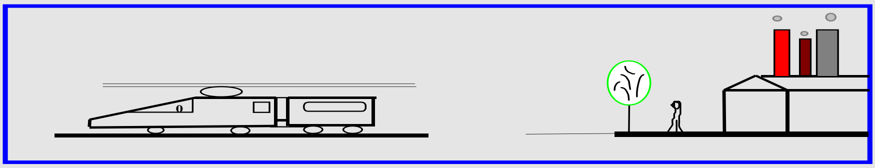
performance standards, with regard both to the operational cost-effectiveness and efficiency and to travelling quality and safety, the analysis of the relations established between the elements which form the characteristic quadrinomial "human being-vehicle-infrastructure-environment" is of strategic importance.



Human being	Environment	Vehicle-infrastructure system
Driver	*Unfavourable atmospheric conditions*	*Deficiencies in technological works and equipment concerning railway lines and stations*
– Psychophysical problems and/or mistaken assessments during the journey; – non-observance of the signs, limits or rules, etc.; – . . .	– Snow, ice, frost, turbulence, fog, etc.; – excessive temperature ranges; – . . .	– Technological anomalies of the fixed equipment – inadequacy of the movable equipment as to the typological conditions of the line; – . . .
Authorized staff and operators	*Calamitous and unforeseen events*	*Problems of uneven and damaged road surface*
– Maintenance deficits – deficiency of operational procedures and actions related (non-observance of rules, regulations, etc.); – . . .	– Telluric movements, landslips, landslides, falling rocks, etc.; – Presence of obstructions on the line, also because of gusts of wind or whirlwinds, etc.; – . . .	– Problems linked to the building typologies of manufactured products, etc.; – lack of the pertinent maintenance protocol, etc; – . . .
Passengers	*Fires in the surrounding territory and other external causes*	*Working problems of movable and fixed equipment*
– Consequences derived from carelessness, sabotage, etc.; – transport of dangerous substances; – . . .	– Specific vulnerability of the territory, etc.; – Pedology and agronomical management of the soils, etc.; – . . .	– Shortage of movable equipment; – reliability of the components of the equipment and signal systems for line spacers, etc.;. – . . .

Fig.1 Risk factors for comfort and operational safety in the railway case

Notably in the planning and management of railways, the problem becomes much more complex, as the bounded guide doesn't admit the continuous and immediate intervention of the driver, in the different elements of the plan - altimetry layout, to balance whit an adequate guide behaviour the effects due to dynamic stresses on the vehicle for complex interactions between infrastructure, vehicle and environment.

In other words the dynamic actions on the vehicle depend only partially by the behaviour of drivers and much more by the line characteristics and so by maintenance actions and by the observance of normative and regulations(1).

However, the problem to guarantee the safety with the condition of psycho - physic comfort of users during the travel, permits, whit the aid of the study of dynamic equilibrium of the vehicle, to determine appropriate standards of global comfort and of safety in exercise condition. Of course in the problem of quality in design's solution is need to consider other factors not negligible, as the dynamic behaviour of the vehicles, the characteristics of the railway and the quality of travel even in presence of anomaly and vibration movements.

Therefore, the research for the optimum solution cannot be shown by a more or less complex mathematical formula, but it should be determined individually, as the case may be, and taking into consideration one or more alternative macro solutions, each of them properly analysed in its conditions, and then modified and further on specifically adjusted, in order to attain the most globally appropriate design solution.

From a methodological viewpoint, therefore, an appropriate model can be constructed, an interpretative model of the interactions among the circulatory phenomenon (binomial "human being-vehicle"), the network component involved (branch, junction, fixed equipment, etc.) and the environment, in order to fully describe the connections between the different variables which characterize both the system reliability as a whole and its compatibilities with the territory and the operational efficiency and cost-effectiveness of the network involved, in order to pre-estimate the *global quality* of the planned intervention.

The Figure 1 shows, for the railway's case, some classes of indicators designed for the analysis of the potential risk factors for comfort and safety concerning the characteristic factor "human being" as well as external causes and a number of problems which can be due to the way and to circumstances of different nature.

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(1) It is to observe that the configurations of dynamic equilibrium in practice depend by stresses between trolleys and bogie of the different elements of the train and by induced system of displacements – rotations on vertical and horizontal planes.

2. *Operational global comfort: a methodological approach for the mathematical characterization of the problem*

The research for designing solutions suited to guarantee acceptable travelling safety and quality standards between two of any relationship origin-destination is one of the key-objectives which needs to be achieved whenever an intervention in a transport network is being carried out. In the railway case, the plain-altimetry option of the axis, the values given to the geometrical elements and the coordination between the horizontal and vertical development of the plan represent strategic variables in order to reach the complete optimisation of the planned design action. Indeed in operation, these particular configurations of the way, together with the peculiarities of the environment pertaining to the infrastructure under consideration and with the specific performance and efficiency properties of the vehicle produce specific "outputs" in the driver during the journey along the pre-established plan, yield consequential travelling quality levels also for passengers.

The concept of "global comfort", implying the safety concept, can be therefore qualitatively formalized as a particular psychophysical condition in which a human being expresses his/her satisfactory relationship with the environment. Basically, it is a condition which cannot be assessed only in the field of the simple physical interactions among human being, vehicle, road and environment, all variables which are the topic of the present study, but it needs to be characterized together with a number of psycho emotional interactions which take place in every single person in a moving condition, by means of his/her own perceptive and sensory systems and personal elaboration of the received inputs. Which follows from the abovementioned definition of comfort that necessarily postulates a psycho emotional imbalance with an advantageous position for a human being.

The same approach provided for the definition of the problem contains the study model which could be referred to mathematically: it is a particular operating system K suited to define the functional relationship between the values characterizing the infrastructure-vehicle-environment trinomial, which are to be seen as stimulus for the user, and the registered condition of global comfort; the latter, in turn, represents the system output analysed as a whole. The suggested analysis scheme is shown in Figure 2.

According to what has been said so far, the problem of investigating the conditions which can produce the comfort state for a user travelling at the speed V along a prearranged plan, with fixed operational configurations and a given vehicle, can be solved by researching the sets of the values which concern the elements of the system described above.

In operational terms, it is then required to analyse the set of the discomfort conditions which can be induced in a human being from the various effects caused, during the journey, by all the elements forming the trinomial “vehicle-infrastructure-environment”.

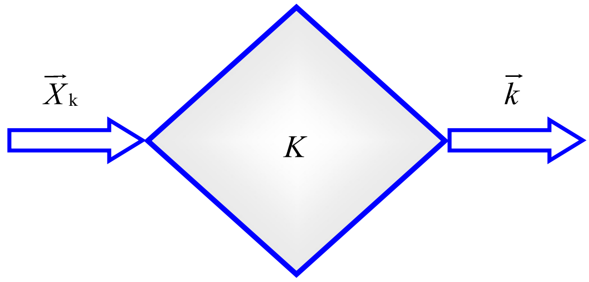

FIG. 2 Operational scheme of the relationship between the algebraic vector $\vec{X}_k$ *of the stimulating values and the induced sensation* $\vec{K}$

Seeing that a state without discomfort is due to those values which characterize such a trinomial and have values falling within the user’s ***domain of biological tolerance***, it follows that the state of absence from discomfort is observed as a consequence of the user’s non-performance of voluntary self-protective actions during the journey.

As a consequence, the group of the states of satisfaction towards the travelling quality standards expressed by a road or railway user can be seen as a particular subset of the states without discomfort. Thus, it follows that ‘not to generate discomfort’ is the necessary condition to experience a comfort state during the journey (Fig.3).

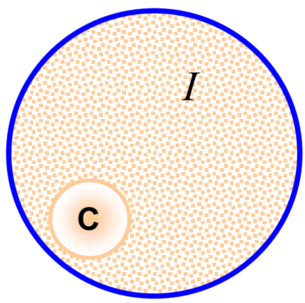

FIG. 3 I denotes the set of the states without discomfort; $C \subset I$ *is the subset of the comfort configurations*

According to such a definition, therefore, comfort can be seen as a subjective and logical value of the “human being system”; in ***Boolean*** terms, then, the value 1 or 0 can be given to the different configurations which take place in operation (the comfort state exists or it does not exist).[(2)]

If we then consider that in the present study several sensory organs are involved simultaneously, the condition of "global comfort" will be consequently obtained by means of a logical product and linked up to the concurrent presence of the comfort states concerning every single sector interested.

In analytical terms, indicating the *global comfort* with C_g and the generic sensory class with the symbol j which defines the *relative comfort* indicator C_j among the d_s globally considered, it follows that:

$$C_g = C_1 \wedge \dots \wedge C_j \wedge \dots \wedge C_d \qquad (1)$$

In order to fully define the problem, a human being can be considered as a system which continually interacts with the surrounding space if we look at the sensations s/he experiences by means of his/her own perceptive and sensory organs (seen as properties of the *boundary* of this set), while environment is meant to denote the set of natural or artificial elements with which a human being coexists and makes exchanges of different entity and typology (including, in such a broader meaning, the infrastructure and the vehicle).

A human being, then, may be assimilated to a system *S* which can be represented by means of appropriate coordinates in $\mathfrak{R}^d$; in such a system, however, the coexistence of the occurring balances (mechanical, chemical, thermal, etc..) is a necessary but inadequate condition to determine a state of global balance. Indeed, the observation of variables of the environment system *A*, as above defined, can induce in human beings such psycho-emotional sensations to urge him/her to react and thus, in order to define the balance of the interactive relationship it is necessary to consider an appropriate interpretative model of the reality which is being examined here.

Consequently, let an appropriate variable U be considered which takes into due account the user's actions of adjustment and behavioural change following his/her interactions with the aforesaid system *A*.

Named $z_1,\dots z_h,\dots z_n$, then, the different sensations which play a role of independent variables of the system *S*, there will be a function so that

$$u = f\,(z_1, \dots, z_h, \dots, z_n); \qquad (2)$$

moreover, the achievement of the balance entails the existence of a particular function **φ** which satisfies the following relationship

$$u = \varphi\,(\alpha_1, \dots, \alpha_k, \dots, \alpha_m); \qquad (3)$$

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(2) For the abovementioned reasons, it is worth noting that, as regards comfort, only a scale of evaluation can be fixed, but not a unit.

being α_1, ..., α_k, ..., α_m the independent variables of the vehicle-infrastructure-environment system.

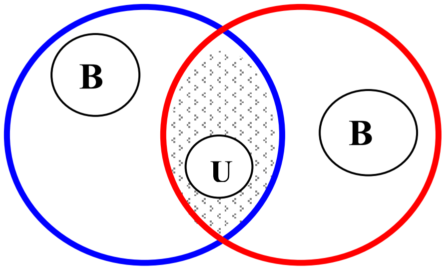

FIG. 4 Set U of the values assumed by the varia ble u and obtained as an intersection of the sets of the values of the state indicato rs Bs in a human being w ith those B $_A$ concerning the vehicle-infrastructure-environment system.

From above, considering (2) and (3), it follows that

$$\Phi\,(z_1, ..., z_n, \alpha_1, ..., \alpha_m) = 0 \qquad (4)$$

The figure (4) shows that the set $\mathbf{B_s}$ of the travelling user's state variables and the set $\mathbf{B_A}$ which characterizes the vehicle-infrastructure-environment trinomial, are composed of elements which lead to a subset intersection **U**[3].

Now, let us observe the domains Z = (z_1, ..., z_h, ..., z_n) and A = (α_1, ..., α_k, ..., α_m) within which the above considered functions f and φ are defined[4], and the several interactions which take place in operation between the system **S**, i.e. road or railway user, and the system **A**, composed of all the elements which are external to the boundary of **S** and interfere with it, given

$$\mathbf{Z} = \mathbf{Z}_v \cup \mathbf{Z}_t \cup \mathbf{Z}_a \cup \mathbf{Z}_h \cup \ldots,$$

where the symbols, v, t, a, h, …, denote respectively the visual, thermal, acoustic, hygrometric, etc.. interactions, we obtain the following relation:

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[3] Similarly, it happens for instance to the temperature in the thermal equilibrium.

[4] If we consider, for example, the set of the values of the variables characterizing the interactions which take place in a moving condition (dynamic perspective vision [ ] [ ]), the relevant sensory duct enables the person concerned to have a one-way correspondence between the environmental values α_{1v}, ..., α_{hv}, ..., α_{nv} and the induced sensation $Z^{\tau}(v)$ which is registered at each moment $\tau \in [0,T]$, where T is the analysis period under consideration. Named $\mathbf{A}_v$, then, the set of values α_{1v}, ..., α_{nv} and Z_v the set resulting from the values of the visual sensation, it follows that: $\mathbf{A}_v \rightarrow \mathbf{Z}_v \rightarrow \mathbf{U}$.

$$Z \rightarrow U^{(5)}$$

As a consequence, the set

$$A = \alpha_v \cup \alpha_t \cup \alpha_a \cup \alpha_h \cup \ldots,$$

in turn will result in a specific variable u characterizing the system A, with $u_i \subset U$.

3. *The operational global comfort vector*

The relations between the elements of the human being-vehicle-infrastructure-environment quadrinomial resulting in (4), define the comfort states globally appreciated by the user during his/her journey within the hyperspace $\Re^d$ of the variables $z_1, .., z_n, \alpha_1, .., \alpha_m$.

The achievement of any comfortable psychophysical sensation C_j (relative comfort) entails the occurrence of the following relation

$$C_j \in \left[C^i{}_j, C^s{}_j \right],$$

where the superscript i denotes the inferior threshold of the comfort state and s the superior one, supposing that every jth sensory class examined assumes an appropriate interval Δ=(s-i) which characterizes the existence field [(6)] of the

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[(5)] Such one-way correspondence between the abovementioned values is given by the function f of (2).

[(6)] For example, examining the aim on the travel comfort and on the same safety performance by the wheel - rail system, it will be need to evaluate the dumping of interaction between rail-superstructure and passenger or freight vehicle. It will too need to evaluate the actions of a running rail vehicle on the line as for example:

- Jerks of trucks idling;
- Tangential and vertical stresses;
- Imperfections and irregularities (micro e macro);
- Suspensions (vertical, transversal, longitudinal);
- Characteristics of the line and superstructure.

aforesaid operational relative comfort state.

Stated $Z_\Delta = (Z^s - Z^i)$ and $\alpha_\Delta = (\alpha^s - \alpha^i)$, let us now construct some particular *reduced variables* by transforming, geometrically, the coordinates of the points in the space $\Re^d$

$$\tau_1 = \frac{z_1 - z^i{}_1}{z_\Delta},\cdots,\ \tau_n = \frac{z_n - z^i{}_n}{z_\Delta}\ ; \qquad (6)$$

$$\varsigma_1 = \frac{\alpha_1 - \alpha^i{}_1}{\alpha_\Delta},\cdots,\varsigma_m = \frac{\alpha_m - \alpha^i{}_m}{\alpha_\Delta} \qquad (7)$$

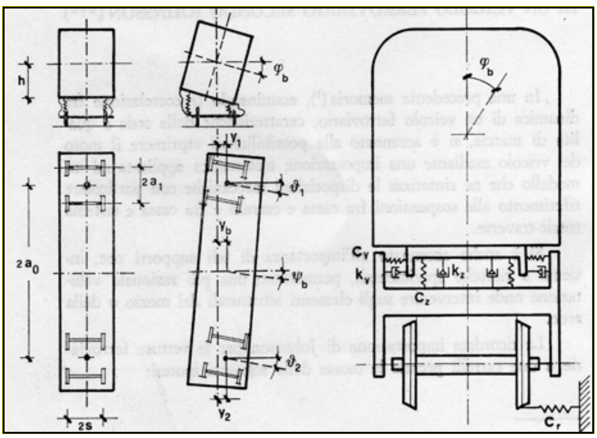
By virtue of (6) and (7), (4) turns to

$$F(\tau_1, \ldots, \tau_v, \varsigma_1, \ldots, \varsigma_m) = 0,$$

locating a surface in the new reduced space $\Re^d{}_2$ which appears to be defined as to axes which are each other *dimensionally homogeneous* (as it happens in the Euclidean space).

In it, the level of the inferior (reduced) comfort threshold corresponds to the origin 0 of the axes, while the points P_s describing the superior threshold belong to a hypersphere of equation

$$\tau_1^2 + \cdots + \tau_n^2 + \varsigma_1^2 + \cdots + \varsigma_m^2 = 1\,. \qquad (8)$$



Those indicators, as happens in the beyond scheme, always admit a value range Δ=(s-i) in which is verified (whit a different quality level) the existence of comfort status for the railway transport users.

Thus, if P is the point which geometrically represents a generic comfort state of the user under consideration, the oriented segment OP$\in$[0, 1] forms in $\Re^k_m$ the measure of the distance for that specific configuration from the inferior comfort threshold, referred to the entire interval of variation Δ=(s-i) observed.

Finally, if as regards the aforesaid reference axes we can denote as $\vec{c}$ the global comfort vector in $\Re^d_r$, geometrically determined by OP, with the following components

$$\vec{c} = \vec{\tau} + \vec{\varsigma}, \qquad \text{with} \qquad \begin{cases} \vec{\tau} = \vec{\tau}_1 + \cdots + \vec{\tau}_n \\ \vec{\varsigma} = \vec{\varsigma}_1 + \cdots + \vec{\varsigma}_m \end{cases}$$

whose modulus $\left|\vec{c}\right|$, according to (8), will be always ≤ 1.

The specific option made with (6) and (7) to define $\Re^d_r$, thanks to the properties of homogeneity which characterize the reduced space built by transforming $\Re^d$, allows to achieve the vectorial expression of (1) as follows:

$$\vec{C}_g \equiv \left\{\vec{C}_i\right\}_1^d \qquad (1').$$

Therefore, the global comfort[(7)] can be expressed by the vector $\vec{C}_g$, having as components the single vectors $\vec{C}_i$ (i=1,…,d) which define in $\Re^d_r$ the specific comfort states relevant to each of the sensory classes considered for the analysis of the travelling quality and safety in the research for the optimisation of the road or railway intervention which is being examined.

For example, in the study of dynamic actions between vehicle and infrastructure, it can be refer for modifications of indicators B_A of system “vehicle – infrastructure – environment” to jerks of trucks idling (Fug 5) due to following circumstances of particular interest for rail practice:

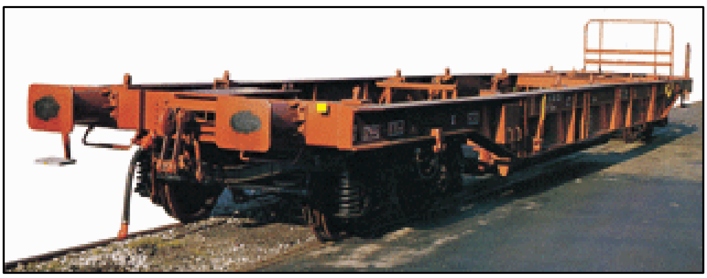


FIG. 5 The jerks of trucks idling induces actions on superstructure whit dangerous effects on the platform due also to dynamic stresses.

[(7)]The peculiar definition of comfort state, given in the previous paragraph 2, allows not to introduce specific weight coefficients δ_i into this paper, which may denote the different importance presumably attached by users to each of the sensory classes.

• Increase UE maximum and commercial speed;
• Increase load/axis from 22,5 t/axis to 25 t/axis ÷ 30 t/axis,
• Increase ratio loaded/empty (whit solutions suitable for reduce the truck load).

The study of global dynamic process could give, as waiting results, to find specific technologies suitable to reduce, in UE sphere, the strains wheel/rail by means of appropriate dumping systems.

4. The territorial information data in the decision support system (DSS)

The problem of the optimal realization of an integrated interventions plan on the rail network can be represented by means of discrete or continuous decision variables, according to the assumptions concerning the discrete quantification of the improvements relating to the single arteries of the system in examination.

In a discrete formulation, a permissible set of change interventions of the geometric-functional variables of the assessed sections is determined, so that the system performances (time or total cost of distance, injurious emissions, etc.) are completely optimal, meeting the requirements of the problem surroundings, like for example, the tie concerning management and investment costs.

The reference logical-structural scheme of DSS is that one shown in figure6; it includes:

- the information system, constituted by a descriptive data base and a geographic-territorial data base and also by the respective management system;
- the models system, constituted by a models base and the respective management system;
- the problems processing module considered like "inner interface";
- the user interface module, famous in literature also like module for the dialogue generation.

For an effective use of the resources of the various components, we must be very careful in the integration process between the systems, both in logical-functional and operating terms.

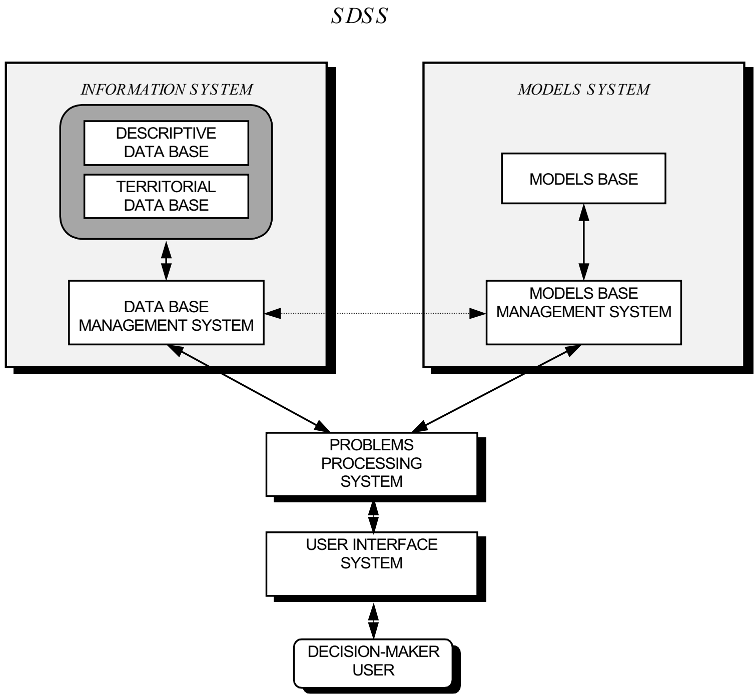


Figure 6 – DSS logical-structural scheme

As for the reference hardware/software architecture for the decision support system, in order to assure the maximum transferability, and therefore the maximum potential diffusion of the application, it’s convenient to take up wide diffusion base hardware/software platforms, but at the same time with middle/high capability of calculation and graphical processing.

We have to consider, moreover, the decision support system in discussion, since finalized to the processing of problems concerning the road system, can be classified like SDSS (Spatial DSS).

In fact, it shows both the characteristics peculiar to a DSS as for the management of a models base and a data one with user interface similar to the user-friendly type and those of a GIS (Geographic Information System) with regard to the management and processing of territorial data.

Because of the particular type of problems to face, this contemporary presence leads, however, beyond the simple sum of the typical functionalities of the two systems, causing an integrated system without any clear distinction between the two original systems.

It is, in fact, possible to work at the same time with the geographic entities (geometry of tracings, intersections, etc.) and with the not closely territorial information related to them, without perceiving operating discontinuity. In such a way, the distinction between territorial data base and descriptive data base is attenuated, assuming a more logical than structural connotation.

The user interface has been developed to execute the maximum integration among the elements of the system and to facilitate the process of user-system "dialogue".

5. Conclusions

The main objective of a railway design is to guarantee the satisfaction of mobility demand of people and goods, assuring regular and safe traffic flow, by providing the infrastructure with appropriate performance and geometrical characteristics (including adequate equipment, road signs, etc.).

The scientific research in the field of transport infrastructures, dealing with safety and quality of the flow, has increasingly suggested to put the user's behaviour (especially in non-guided ways) at the centre of the technical design options, in building new works as well as in carrying out further interventions in the network (functional improvement, calibration of interchanges, etc.)

The study of the system "human being-vehicle-infrastructure-environment" carried out in this paper has allowed to define an appropriate mathematical model for the integrated analysis of the various components which characterize the user's travelling quality and also influence the operational safety.

The mathematical approach of the problem we have suggested, which refers to set theory and vector calculus, allows an appropriate geometrical interpretation in a hyperspace of d-dimension and is also suitable for computerized processing.

Notably the suggested criterion which makes use, in $\Re^d$, of the analysis of the **operational global comfort vector** $\vec{C}_g^i$ constructed for each itch alternative of intervention examined, allows to acquire useful information in order to accurately select the best design solution. Therefore, in order to guarantee a satisfactory travelling quality to users and to increase the safety standards concomitantly, it will be better to adopt that specific option h which satisfies the relation

$$\left|\vec{C}_g^h\right| = \max\left\{\left|\vec{C}_g^i\right|\right\}$$

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RENDICONTI DEL CIRCOLO MATEMATICO DI PALERMO
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A S.U.R.E. Estimation Procedure of Italian Intra-Household Working-Time Allocation

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Summary

The present analysis demonstrates how it is possible to obtain a simultaneous estimate of both domestic and external working time by taking into account the influence of the family typology. Both the family context and a gender component influence the allocation of working-time. As a gender component is not directly observable, a SURE-GLS procedure is adopted. The estimation results show how the subject that, after the dissolution of his previous union reconstitute a new family, has even a greater bargaining power in the division of working-time with the new partner. Those who have reconstituted a new family, finally, spend more time in market-working activity .

1. Introduction

The analysis of the allocation of the time in Italy has received special impulse in recent times contextually to the surveys conducted by Istat and, in part, by the Bank of Italy[1]. The empirical results of these surveys show, among the other things, the strong impact on the allocation of the time of a marked gender component, especially in the family of traditional type (Sabbadini-Palomba, 1994 and Menniti-Palomba, 2001). Working-time allocation is also an important factor in the estimates of the household production and of the indirect cost of the children (Di Pino, 2004, Di Pino-Mucciardi, 2004). Working-time allocation between external work, domestic work and, eventually, study is explained by a series of socio-demographic information indicative of the family context of the subject. From the methodological point of view, we propose to estimate a "Seemingly Unrelated Regression Equations" model *(SURE),* where the dependent variables are represented by the time individually spent in a week, respectively, at external work, housework and to study. The *SURE* model estimation is supported by a series of

[1] Cfr. Istat: Multipurpose Survey on Families and Social Subjects and Time Use Survey; Bank of Italy: Survey on Households Income and Wealth (*SHIW*).

preliminary "reduced-form" estimates of some endogenous variables included in the model such as the information about the typology of the family of the subject, the education and the probability to enter the labour market. The reduced-form preliminary estimates give instrumental variables that we utilize in the model in substitution of the previous potentially endogenous factors. The specification of the model is based on two hypotheses, supported by some descriptive analysis conducted preliminarily: 1) that the gender component influences jointly the allocation of the time between external work, domestic work and study (especially the first two); 2) that in the regions of the North of Italy the data regarding the working time are characterized by a hierarchical structure (individual on the first level, family on the second level); this structure is absent, instead, in the other areas. We may hypothesize that the families of the North show a greater degree of "collegiality" in the decisions about the time that each individual can spend at work. It's necessary to emphasize, nevertheless, that the decisions of the family's components to work can be strongly influenced by external factors such as employment opportunities, quality of social services and flexibility in the working hours that, as we know, differ significantly at the local level. In the specification of the model we take into account the structural differences that there exist in the three geographical areas of Italy: North, Centre and South. Furthermore, we include in the specification of the model the simultaneous impact of a gender component on the partners' behaviour by modifying the **SURE** estimation procedure, here proposed in a **GLS three-stage** version (Srivastava-Giles, 1987). Practically, we assume that a gender factor is a unobservable variable that influences simultaneously men and women**.** We try therefore to capture this influence through a particular specification of the errors. The plan of work is the following: in the second paragraph, after a brief survey of the literature on microeconomic and econometric foundations of the model, the estimation methodology is proposed; in the third the results of the estimation and simulations are discussed.

2. Specification of a working-time model.

Traditionally, the working time allocation is specified by using a system of simultaneous equations, where each equation explain the behaviour of one of the partners (Lundberg, 1988; Fortin-Lacroix, 1997; Chiuri, 1999). The dependent variables are usually represented by both the working time spent in the labour market and the time devoted to domestic activity. Sometimes, the system is specified only by two simultaneous equations of the working time of the two partners, while the domestic work is classified generically as "nonlabour income" and is utilized as explanatory variable, with evident problems of endogeneity and inconsistency of estimates and hypothesis testing results. The dependent variables are usually represented by both the working time spent in the labour market and the time devoted to domestic activity. The estimation of our model allows us to verify specific behavioural assumptions. For example, if the difference between the coefficients of household income and the coefficient of the wage of the subject is not significant, it shapes a traditional family model, where the behaviour of spouses about the allocation of the resources within the family is characterized by absolute altruism (Becker, 1991). Other tests on specific parametric constraints may lead, instead, to accept or reject the hypothesis that the two partners, even if not completely altruist, succeed however to bargain about a division of the working time that, if really non satisfactory for both, is at least “Pareto-efficient" (in the sense that every different allocation of the time would leave one or both partners still more unhappy) (Manser-Brown, 1980, Chiappori, 1988, Browning-Chiappori, 1998). Indeed, the assumption that the behaviour of each subject is characterized by altruism is weakened when it takes into account the possibility that the union can dissolve. The eventuality (even distant) of a divorce, in fact, and the forecasts of the connected "costs" can constitute, in itself, an additional factor that influences the "bargaining" on the allocation resources within family (Becker, Landes, Michael, 1977; Weiss, 1997; Chiappori, Fortin, Lacroix, 2001).

In this sense, we may reasonably believe that the behaviour of a subject in the labour framework won’t influence his choice of life about the constitution or the break-up of the union, or in terms of reproductive behaviour. All this, however, involves further complications both for the model specification, and for the use of the estimation methodology, which must take into account the risk that the estimation results are inconsistent owing to the presence of endogeneity in the relationship between work and demographic variables.

We start from the assumption that allocation of time of family components and “demographic” factors influence each other. Therefore, to obtain estimates corrected by endogeneity we create preliminarily four instrumental variables (**IV**) in substitution of the correspondent potentially endogenous variables, that are: 1) presence of the partner, 2) education level, 3) separated or divorced, 4) participation in the labour market. The residuals of the preliminary **IV** estimates provide the correction instruments for endogenous relationships in the equations of the model (Heckman, 1976).

The simultaneous estimation of the market working time ($\mathbf{H_l}$), of the domestic working time ($\mathbf{H_f}$), and of the time dedicated to study ($\mathbf{H_s}$) is given by the **SURE-GLS** estimation of the following system of equations:

$$H_l = \alpha_0 + \alpha_1 CH + \alpha_2 edu^* + \;_3\;^{*}_{1} + \alpha_4 \lambda_4^* + \alpha_5 H_{lP} + \\ + \alpha_6 H_{fP} + \alpha_7 RW + \alpha_8 N_{P=1} + \alpha_9 N_{P=0} + \alpha_{10} DIV + \alpha_{11} \lambda_3^* + \varepsilon_l \tag{1.1}$$

$$H_f = \beta_0 + \beta_1 CH + \beta_2 edu^* + \beta_3 \lambda_1^* + \beta_4 \lambda_4^* + \beta_5 H_{lP} + \\ + \beta_6 H_{fP} + \beta_7 RW + \varepsilon_f \tag{1.2}$$

$$H_s = \gamma_0 + \gamma_1 CH + \gamma_2 edu^* + \gamma_3 \lambda_1^* + \gamma_4 \lambda_4^* + \gamma_5 H_{lP} + \\ + \gamma_6 H_{fP} + \gamma_7 H_{sP} + \gamma_8 DIV + \gamma_9 \lambda_3^* + \gamma_{10} RW + \varepsilon_s \tag{1.3}$$

The dependent variables of the equations of the model are represented, respectively, by the time dedicated weekly to external work (1.1), to domestic work (1.2) and to study (1.3). The explanatory variables are the following: the instrumental variables generated from the first four equations, the number of children with age 0-5 years (CH) that are present in the family, the time (external work H_{Pl} and domestic H_{Pf}) dedicated by the partner to external work, H_{Pl}, to domestic work, H_{Pf}, and to study (H_{Ps}),the reservation wage (RW), and the ratio between the number of active members (in labour market) and the number of the components in the family ($N_{P=1}$ with the partner and $N_{P=0}$ without partner) as a proxy of the job opportunities for the family in the socio-economic context. The specification of the $SURE$ equation system is based on the assumption that the dependent variables, or the quantity of time that each subject spends, respectively, in market working, in domestic activity and to study, are jointly correlated and depending on some common factors, generally unobservable. The joining correlation among dependent variables may be formalized by modelling the relationships between error terms of the three equations of the model ($\varepsilon_l, \varepsilon_f, \varepsilon_s$). In the specific case an additional assumption is introduced in the specification of the error terms: we assume that their relationships is influenced by fixed effects correlated to the unobservable gender component. As a consequence of this, the following specification of the relationships between the errors of the previous equations is given:

$$\varepsilon_l = \phi_0 + \phi_{1m} \varepsilon_{fM} + \phi_{1w} \varepsilon_{fW} + \phi_{2m} \varepsilon_{sM} + \phi_{2w} \varepsilon_{fW} + \phi_3 DUM.MW + \eta_l; \tag{2.1}$$

$$\varepsilon_f = \varphi_0 + \varphi_{1m} \varepsilon_{lM} + \varphi_{1w} \varepsilon_{lW} + \varphi_{2m} \varepsilon_{sM} + \varphi_{2w} \varepsilon_{fW} + \varphi_3 DUM.MW + \eta_f; \tag{2.2}$$

$$\varepsilon_s = \theta_0 + \theta_{1m} \varepsilon_{lM} + \theta_{1w} \varepsilon_{lW} + \theta_{2m} \varepsilon_{fM} + \theta_{2w} \varepsilon_{fW} + \theta_3 DUM.MW + \eta_s; \tag{2.3}$$

In previous equations the **m** and **w** suffixes show that for each observation the sex of the subject (man or woman) changes the structural relation among the errors, so that the dummy variable **DUMMW** (man = 0; woman = 1) extends the hypothesis of structural change even to the constant terms. In short, the relations among the errors are modelled hypothesizing a global structural change, both in the slopes and in the intercept of each equation. Theoretically this means that belonging to the male or female sex determines the unobservable component of the time allocation influencing the reciprocal relations among different methods of use of time. Previous assumptions on the relationships among the errors, obviously, involve an adaptation of the estimation methodology that, in this particular case, is represented by a GLS **Three Stage** procedure (**3S-GLS**) [2]. With regard to the new typologies of family, the model contains explanatory variables that signal if the subject is part of a reconstituted family or not. If we consider the status of the subject inside the family potentially endogenous with respect to the working-time, we will use in the estimates appropriate instrumental variables, referred to the residuals ($\lambda^*_1, \lambda^*_3, \lambda^*_4$) of the preliminary **IV** estimations (see Appendix) that contribute to correct the effects of the endogeneity.

3. The data and the estimation results

The empirical sample is drawn from the data of the Istat-Multipurpose survey conducted in 2003 (FSS03). The sample size consists of 40.972 individuals aged over 18 years. The existence of a significant hierarchical structure in the data regarding the hours of work in the North of Italy (not found in the other areas of the country) involves different estimates in the three macro-areas North, Center and South. The relative three sub-sample are respectively of 20232 individuals included in 8355 families in the regions of the North, of 9015 individuals distributed in 3626 families in the Center and of 20294 individuals in 7246 families in the Southern regions. Table 1 (see Appendix) shows the results of the Instrumental Variables estimates (the equations 1 - 4); Table 2 and Table 3 show the estimation results at the third stage of the time allocation between external work, domestic work and study (system of **SURE** equations), respectively, in the North and in the South of

[2] The 3S-GLS estimator involves a Three-Stage procedure. At the first stage the Equations of the GLS model specification are estimated by **OLS**. The residuals of the first-stage of each equation are utilized, at the second stage, as dependent variable in a **OLS** regression on the residuals of the other equations. Finally, at the third stage, the **OLS** estimation of the first stage is repeated with the previous application of a correction to the dependent variables, to whom are subtracted the residuals "explained" by the second-stage regressions.

Italy. Regarding the estimate of the market working time in the North (Table 2 in Appendix), it results how the time dedicated to external work is lower for the families with children. Furthermore, the external work of the subjects who live in couple show a significant inverse correlation with the external work of the partner and a contextual significant direct correlation with the domestic work of the partner. These two results in particular seem compatible, in a traditional family context, with the assumption of specialization of the roles of the partners. However, the estimates show that persons who dedicate more time to external work normally are members of families with a larger number of active components on the labour market, including partner. The latter result seems to configure, instead, a different behaviour of family components with respect to the hypothesis of specialization. In the North of Italy, the results of the estimate of the domestic work show the existence of a significant direct relationship between time devoted to domestic activity by the subject and external work of the partner. That is, obviously, compatible with the hypothesis of specialization. Similar results, regarding the relationships of the working times with those of the partner, derive from the estimates in the South of Italy and confirm even in this context the hypothesis of the existence of the specialization of the roles (Table 3 in Appendix). The results of estimates and simulations of the model (Table 4 in Appendix) show, however, that a tendency towards the "complete specialization" of the role between partners doesn't exist, since both often result contemporarily present on the labour market, above all in the North. It may be interesting to notice how the residuals of the estimated probability to live in couple (λ_1^*), that represent a proxy of the costs of searching for a new partner, produce a negative influence on the external working time and positive influence on domestic work of the subject. Therefore, we can assume that – in the perspective of dissolution of the actual union – the subject that could more easily find a new partner has a greater bargaining power inside the couple, and can impose a division of the working times that leaves to the other partner a bigger charge of domestic work. The results of the simulations in Northern regions show that those who have experimented the separation spend more time in external work if have reconstituted a new family.

4. Final observations

The results here obtained in any case have to find confirmation in analysis conducted with different or alternative specifications of the model. In particular, it would be useful to compare the result of this analysis with an other, analogous, that may be conducted on data that show the employment of the time with greater rigor and specificity. Moreover, it would be interesting to propose the estimation of an analogous model using data set that brings information even on incomes and consumptions of individuals and families[3]. Regarding some of the many questions left opened by the present analysis, the problem about how to treat in the model the time dedicated to the study seems interesting. That is, it is possible to assume the study as an employment of time alternative to work (domestic and external) and, consequently, estimate it simultaneously in a SURE model, or treat it as explanatory variable (obviously endogenous) and estimate it apart as instrumental variable. The choice between these two approaches depends greatly on the characteristics of empirical information in the sample.

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Appendix – Estimation Results

Tab. 1 - Instrumental Variables Estimates

	Eq. (1)		Eq. (2)		Eq. (3)		Eq. (4)	
Estimator	*Probit*		*OLS*		*Probit*		*Probit*	
dependent variable	Partner		Ln(Istr)		Div		Work	
explanatory variables								
constant term	-1.3354	(*)	1.3013	(*)	-0.5372	(*)	-4.1853	(*)
(dummy) sex (0 = Man; 1 =Woman			-0.1070	(*)				
(dummy) Health (0 = bad; 1 = good)			0.6239	(**)				
(dummy) Regional Area			-0.0188	(*)				
sex (ref. man)	-0.0550	(*)			-0.0455	(*)	0.5364	(*)
area (ref. Sud)								
North	0.0320	(*)			0.0832	(*)	0.2689	(*)
Centre	0.0367	(*)			0.0302	(*)	0.1269	(*)
age	0.0465	(*)			-0.0190	(*)	0.2348	(*)
age^2							-0.0027	(*)
parents education					0.0433	(*)	0.0096	(*)
ln(parents education)			0.2139	(*)				
mother education	-0.0355	(*)						
father education	-0.0169	(*)						
(Dummy) religiosity (ref. yes)	0.0684	(*)			-0.1756	(*)		
siblings	0.0438	(*)						
ln(siblings)			-0.0354	(*)				
ln(age parents)			0.0231	(*)				
education* (explained by Eq. 2)					-0.0196	(*)		
λ*1 (Heckman Correction Eq. 1)					-0.9259	(*)		
Adj-R^2			26.00%					
p-value: (*) < 1%; (**) < 5%								

Tab. 2 –North of Italy: Allocation of time estimation results. III stage

Estimator: SURE 3S-GLS - IIstage

dependent variable (weekly hours)	correct market working-time		correct non-market working-time		correct study-time	
explanatory variables						
costant term	37.892	(*)	272.726	(*)	-32.049	(*)
Children 0-5	-27.811	(*)	45.726	(*)	-10.196	(*)
education* (explained Eq. 2)	15.942	(*)	-13.538	(*)	10.318	(*)
l*1 (Heckman Corr. Eq. 1-partner)	-142.988	(*)	52.115	(*)	0,1257	#
l*3 (Heckman Corr. Eq. 3-divorce)	19.408	(*)			-48.539	(*)
l*4 (Heckman Corr. Eq. 4-Work)	279.024	(*)	-124.350	(*)	-68.116	(*)
H_{lP}, external working-hours partner	-0.0188	(*)	0.1899	(*)	-0.0903	(*)
H_{lF}, domestic working-hours partner	0.3082	(*)	-0.0370	(*)	-0.0466	(*)
H_{lS}, study-hours partner					0.1130	(*)
DIV (Dummy divorce: 0 = no; 1 = yes)	57.264	(*)			-73.162	(*)
RW (reservation wage)	0.0019	(*)	-0.0012	(*)		
$N_{P=0}$ (Employed/family members, Partner	168.003	(*)				
$N_{P=1}$ (Employed/family members, Partner	91.565	(*)				
Adj-R^2	65.00%		24.00%		17.00%	

p-value: (*) < 1%; (**) < 5%; # >=5%

Tab. 3 – South of Italy: Allocation of time estimation results. III stage

Estimator: SURE 3S-GLS - II stage

dependent variable (weekly hours)	correct market working time		correct non-market working time		correct study-time	
explanatory variables						
costant term	67.059	(*)	223.269	(*)	-15.850	(*)
CH 0-5	-27.370	(*)	39.305	(*)	-13.785	(*)
education* (explained by Eq. 2)	0.8153	(*)	-0.8901	(*)	12061	(*)
costant term	-0.0003	#	0.0002	(**)	-0.0001	#
Children 0-5	148.553	(*)	438.102		-	(*)
education* (explained Eq. 2)	300.356	(*)	-87.085	(*)	-51.120	(*)
l*1 (Heckman Corr. Eq. 1-partner)	-0.0498	(*)	0.2808	(*)	-0.1066	(*)
l*3 (Heckman Corr. Eq. 3-divorce)	0.3274	(*)	-0.0854	(*)	-0.0731	(*)
l*4 (Heckman Corr. Eq. 4-Work)					0.2110	(*)
DIV (Dummy divorce: 0 = NO; 1 = YES)	177.009	(*)	449.824		-	(*)
RW (Reservation Wage)	0.0007	(*)	-0.0007	(*)	-0.0005	
$N_{P=0}$ (Employed/family members, Partner =no)	113.238	(*)				
$N_{P=1}$ (Employed/family members, Partner =yes)	113.198	(*)				
Adj-R^2	69.62%		24.18%		19.48%	

p-value: (*) < 1%; (**) < 5%; # >=5%

Tab.4 -External and domestic work in different family's typology. Simulating the model's results in the North of Italy

Unmarried single

Man	weekly hours	Woman	weekly hours
External work	24	External work	26
Housework	14	Housework	14
Total	38	Total	40

Divorced single man/woman without children

divorced and single man	weekly hours	divorced and single	weekly hours
External work	36	External work	37
Housework	15	Housework	15
Total	51	Total	52

Divorced single woman with 1 child 0-5

divorced and single	weekly hours
External work	26
Housework	20
Total	45

Traditional family

married man without	weekly hours	married woman without	weekly hours
External work	36	External work	32
Housework	20	Housework	22
Total	56	Total	54
married man with 1 child 0-	weekly hours	married woman with 1	weekly hours
External work	27	External work	24
Housework	25	Housework	26
Total	52	Total	50
married man with 2 children	weekly hours	married woman with 2	weekly hours
External work	22	External work	18
Housework	29	Housework	31
Total	51	Total	49

Recostituted family

divorced man without children	weekly hours	divorced woman without children	weekly hours
External work	42	External work	38
Housework	20	Housework	22
Total	62	Total	60
divorced man with 1 child 0-5	weekly hours	divorced woman with 1	weekly hours
External work	33	External work	30
Housework	25	Housework	26
Total	58	Total	56
divorced man with 2 children	weekly hours	divorced woman with 2	weekly hours
External work	28	External work	24
Housework	29	Housework	31
Total	57	Total	55

RENDICONTI DEL CIRCOLO MATEMATICO DI PALERMO
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A BUFFON TYPE PROBLEM FOR A SMALL PARALLELOGRAM

Andrei DUMA - Sebastiano RIZZO

Abstract

We consider as test body a parallelogram P with sides a and b and an acute angle $\alpha \in \left]0, \frac{\pi}{2}\right]$ and we determine the probability p_s that P, uniformly distributed in a bounded region of the euclidean plane, intersects on a straight line of the lattice R_A of Buffon (shaped by parallel and equidistant straight lines with equidistance A) a segment of length at least equal to a fixed real positive number s.

AMS 2000 Subject Classification : Geometric probability, stochastic geometry, random sets, random convex sets and integral geometry.
AMS Classification : 60D05, 52A22

§ 1 Introduction

Let be given in the euclidean plane a lattice R_A of Buffon (shaped by parallel and equidistant straight lines with equidistance A) and we consider as test body a parallelogram P with sides a and b and an acute angle $\alpha \in \left]0, \frac{\pi}{2}\right]$. We want to determine the probability p_s that P, uniformly distributed in a bounded region of the euclidean plane, intersects on a straight line of the lattice R_A a segment of length at least equal to a fixed real positive number s. It is sufficient to consider only $\alpha \in \left]0, \frac{\pi}{2}\right[$ because the case $\alpha = \frac{\pi}{2}$ was considered in [2].

Let be $b \leq a$, let M be the barycentre of P and let d and D, $d < D$, the two diagonals of P. We denote by β_b (respectively γ_b) the angle between a and D (respectively between a and d) and we denote by β_a (respectively γ_a) the angle between b and D (respectively between b and d). Hence

$$\beta_b + \beta_a = \alpha \quad , \quad \gamma_b + \gamma_a = \pi - \alpha$$

$$\frac{b}{\sin(\beta_b)} = \frac{a}{\sin(\beta_a)} = \frac{D}{\sin(\alpha)} \quad , \quad \frac{b}{\sin(\gamma_b)} = \frac{a}{\sin(\gamma_a)} = \frac{d}{\sin(\alpha)} \, .$$

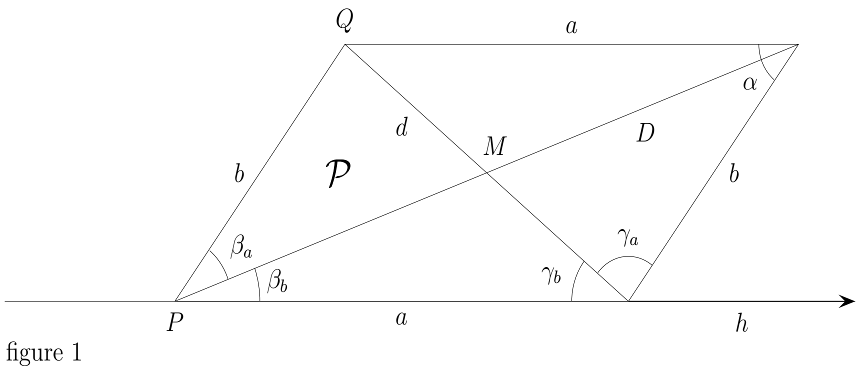

figure 1

Since $\alpha < \frac{\pi}{2}$, we observe $\min\{b\sin(\alpha), b, a\sin(\alpha), a, d, D\} = b\sin(\alpha)$. (Only in the case $\gamma_b = \frac{\pi}{2}$ we have $b\sin(\alpha) = d$.) We suppose that P is "small" compared with the lattice R_A (see [1]) and thus

$$D = \max\{b\sin(\alpha), b, a\sin(\alpha), a, d, D\} < A.$$

We denote by h the oriented straight line that contains one of the sides of P of length a, by g a oriented straight line of the lattice R_A and by φ the angle between g and h. In this way we obtain, except translations, all possible positions of P in respect of R_A if $\varphi \in [0, \pi[$.

To calculate the probability p_s it is sufficient to suppose:

- the barycentre M of P to be in a strip S of width $\frac{A}{2}$ and to have g as one of the two components of the boundary,
- the projection of M to be a fixed M_0 of g,
- the angle φ to be in $[0, \pi[$.

The couple (M, φ) completely characterizes P and we denote (M, φ) the data of P. Now we want to determine the probability p_s for $s \in [0, D]$. The probability p_0 is known (see [3]), $p_0 = \frac{2}{\pi}\frac{a+b}{A}$, and obviously $p_D = 0$. Let $P(\varphi, s)$ be the test body with data $(M(\varphi), \varphi)$ intersecting on g a segment of length s. Let $x(\varphi)$ be the distance $\overline{M(\varphi)M_0}$ that is the distance of $M(\varphi)$ to g; if $M(\varphi)$ does not exist, we put $x(\varphi) = 0$. Then the test body P with data (φ, s) intesects on g a segment of length at least s if and only if M is between $M(\varphi)$ and M_0. Due to a formula of Stoka (see for example [3]) we have

$$p_s = \frac{\int_0^{\pi} x(\varphi)d\varphi}{\int_0^{\pi} \frac{A}{2}d\varphi} = \frac{2}{\pi A}\int_0^{\pi} x(\varphi)d\varphi. \tag{1}$$

To determine $x(\varphi)$ we have to consider the different possible relations between the six relevant lengths $\{b\sin(\alpha), b, a\sin(\alpha), a, d, D\}$ of the parallelogram and thus to consider the different sub-intervals of $[0, D]$ which s can vary in.
In the following section we consider, only for a "small" parallelogram P compared with the lattice R_A, all possible cases.

§ 2 The case that s is smaller than the six lengths of P

Explicitly spoken we consider the "small" case: $s \leq b\sin(\alpha) = \min\{b\sin(\alpha), b, a\sin(\alpha), a, d, D\}$. Except for $\varphi = \frac{\pi}{2}$ and for $s = b\sin(\alpha)$ there exist only a couple of points $A(\varphi)$ on the side a and $B(\varphi)$ on the side b such that the segment $\overline{A(\varphi)B(\varphi)}$ has the length s. The function $\varphi \to x(\varphi)$ takes different shapes if $\varphi \in [0, \pi - \alpha]$ or if $\varphi \in [\pi - \alpha, \pi]$.
If $\varphi \in [0, \pi - \alpha]$, by looking at figure 2 and its notations (The straight line g' is parallel to g but if $\varphi \neq 0$ it is not a straight line of R_A.), we define

$$\left|\overline{M(\varphi)M_0}\right| =: x(\varphi)\ ,\ \left|\overline{M'M_0}\right| =: y(\varphi)\ ,\ \left|\overline{A(\varphi)B(\varphi)}\right| =: s\ ,\ \left|\overline{PA(\varphi)}\right| =: z(\varphi)$$

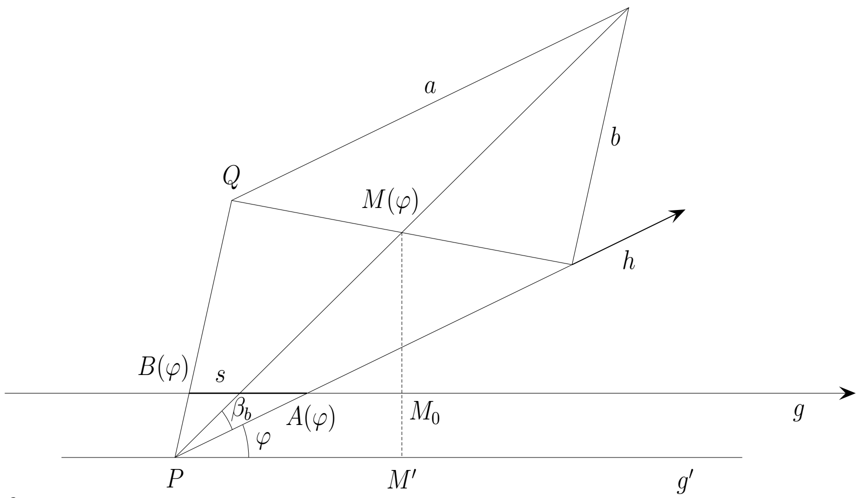

figure 2

and we obtain

$$\frac{z(\varphi)}{\sin(\alpha+\varphi)} = \frac{s}{\sin(\alpha)} \quad , \quad y(\varphi) = z(\varphi)\sin(\varphi) = \frac{s\sin(\alpha+\varphi)\sin(\varphi)}{\sin(\alpha)} \quad , \quad x(\varphi) + y(\varphi) = \frac{D}{2}\sin(\beta_b+\varphi)$$

and thus because of $\sin(\alpha+\varphi)\sin(\varphi) = \frac{1}{2}\left[\cos(\alpha) - \cos(\alpha+2\varphi)\right]$

$$x(\varphi) = \frac{D}{2}\sin(\beta_b+\varphi) - \frac{s}{2}\cot(\alpha) + \frac{s}{2\sin(\alpha)}\cos(\alpha+2\varphi). \tag{2}$$

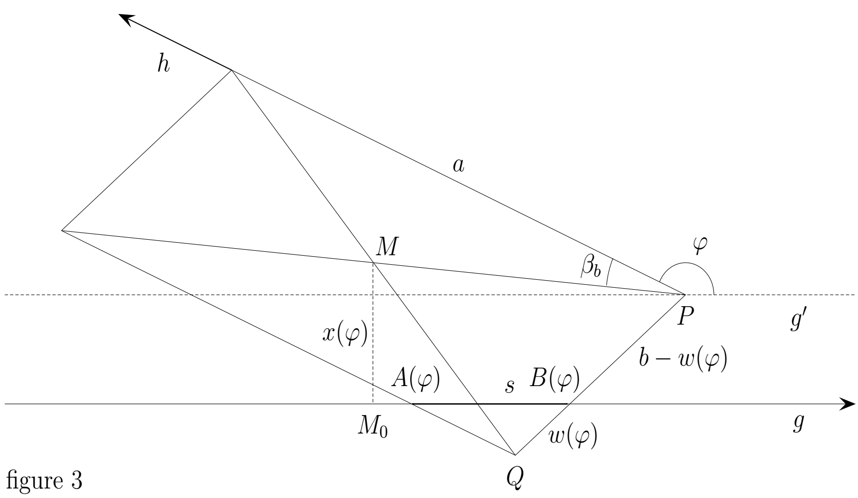

figure 3

For $\varphi \in [\pi-\alpha, \pi-\beta_b]$, by looking at figure 3 and its notations, we get

$$\left|\overline{QB(\varphi)}\right| = w(\varphi) = \frac{s\sin(\varphi)}{\sin(\alpha)} \quad , \quad x(\varphi) = \frac{D}{2}\sin(\pi-\varphi-\beta_b) + \big(b - w(\varphi)\big)\sin(\alpha+\beta-\pi)$$

and thus

$$x(\varphi) = \frac{D}{2}\sin(\beta_b+\varphi) - b\sin(\alpha+\varphi) + \frac{s}{2}\cot(\alpha) - \frac{s}{2\sin(\alpha)}\cos(\alpha+2\varphi). \tag{3}$$

For $\varphi \in [\pi-\beta_b, \pi]$ we obtain, by utilizing figure (3'), the same value $x(\varphi)$ of formula (3) because $\big(b-w(\varphi)\big)\sin(\alpha+\varphi-\pi) - \frac{D}{2}\sin(\varphi+\beta_b-\pi) = \frac{D}{2}\sin(\beta_b+\varphi) - b\sin(\alpha+\varphi) + w(\varphi)\sin(\alpha+\varphi)$.

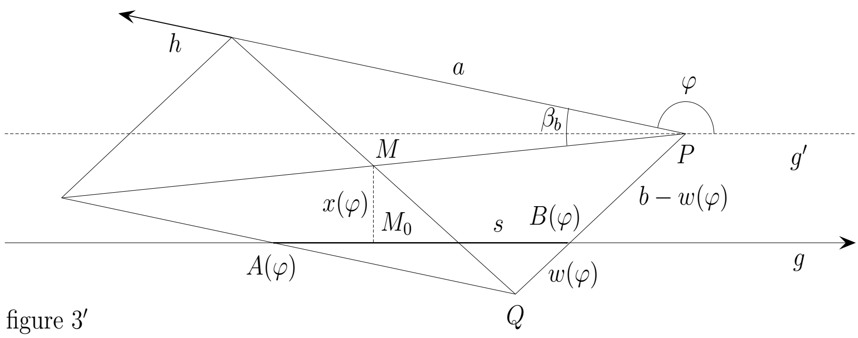

figure 3′

We observe that if $\varphi = \pi - \alpha$ formulas (2) and (3) are equal. Hence

$$\int_0^{\pi} x(\varphi)d\varphi = \int_0^{\pi-\alpha} x(\varphi)d\varphi + \int_{\pi-\alpha}^{\pi} x(\varphi)d\varphi =$$

$$\frac{D}{2}\left[\cos(\beta_b) - \cos(\beta_b+\pi)\right] + \frac{s}{4\sin(\alpha)}\left[\sin(\alpha+2\pi-2\alpha) - \sin(\alpha)\right] - \frac{s}{2}\left[\pi-\alpha\right]\cot(\alpha) + \frac{s}{2}\alpha\cot(\alpha) +$$

$$b\left[\cos(\alpha+\pi) - \cos(\alpha+\pi-\alpha)\right] - \frac{s}{4\sin(\alpha)}\left[\sin(\alpha+2\pi) - \sin(\alpha+2\pi-2\alpha)\right] =$$

$$D\cos(\beta_b) - s - s\big(\frac{\pi}{2}-\alpha\big)\cot(\alpha) + b\left[1-\cos(\alpha)\right].$$

Because $D\cos\beta_b = a + b\cos\alpha$ we have

$$\int_0^{\pi} x(\varphi)d\varphi = a + b - s - s\big(\frac{\pi}{2}-\alpha\big)\cot(\alpha). \tag{4}$$

In this way we have demonstrated the following:

Proposition 1. If $s \le b\sin\alpha$ the required probability p_s is:

$$p_s = \frac{2}{\pi A}\Big\{a + b - s\big[1 + \big(\frac{\pi}{2}-\alpha\big)\cot(\alpha)\big]\Big\}. \tag{5}$$

Remarks:

1) $p_0 = \frac{2}{\pi A}(a+b)$ is the formula demonstrated by Stoka (see [3]).

2) $a+b-s\left[1+(\frac{\pi}{2}-\alpha)\cot(\alpha)\right]$ is a positive number and, because $s \leq b\sin(\alpha)$, we obtain

$$\int_0^\pi x(\varphi)d\varphi = a+b-s-\frac{s}{\sin(\alpha)}\Big(\frac{\pi}{2}-\alpha\Big)\cos(\alpha) \geq a+b-b\sin(\alpha)-b\Big(\frac{\pi}{2}-\alpha\Big)\cos(\alpha).$$

3) The function $h : \left[0, \frac{\pi}{2}\right] \to R$, $h(\alpha) = \sin(\alpha) + \left(\frac{\pi}{2}-\alpha\right)\cos(\alpha)$ obviously has the derivative $h'(\alpha) = -\left(\frac{\pi}{2}-\alpha\right)\sin(\alpha) \leq 0$. Hence $h(\alpha) \leq h(0) = \frac{\pi}{2}$ and thus

$$\int_0^\pi x(\varphi)d\varphi \geq a+b-\frac{\pi}{2}b = a-\Big(\frac{\pi}{2}-1\Big)b > 0$$

since $b \geq a$ and $\frac{\pi}{2}-1 < 0,6$. For $s \leq b\sin(\alpha)$ follows

$$p_s \geq \frac{2}{\pi}\frac{a-b\left(\frac{\pi}{2}-1\right)}{A}.$$

§ 3 The "not small" cases for s.

If $b \leq a\sin(\alpha)$, we have two possibilities:

(I) $b\sin(\alpha) < b \leq a\sin(\alpha) \leq d \leq a \leq D$,

(II) $b\sin(\alpha) < b \leq a\sin(\alpha) < a \leq d < D$.

If $a\sin(\alpha) \leq b$, we have three possibilities:

(III) $b\sin(\alpha) \leq a\sin(\alpha) \leq b \leq a \leq d \leq D$,

(IV) $b\sin(\alpha) \leq a\sin(\alpha) \leq b \leq d \leq a \leq D$,

(V) $b\sin(\alpha) \leq a\sin(\alpha) \leq d \leq b \leq a \leq D$.

The first "not small" case is (A): $b\sin(\alpha) \leq s \leq \min\{b, a\sin(\alpha)\}$.

This case is common to all the situations (I), (II), (III), (IV) and (V). Let $\varphi_1 \in \left[0, \frac{\pi}{2}\right]$ be uniquely determined by the equality $\sin(\varphi_1) = \frac{b\sin(\alpha)}{s}$. Because $\sin(\varphi_1) = \frac{b\sin(\alpha)}{s} \geq \frac{b\sin(\alpha)}{b} = \sin(\alpha)$ and $\alpha < \frac{\pi}{2}$, we obtain $\varphi_1 \geq \alpha$ (see fig.4).

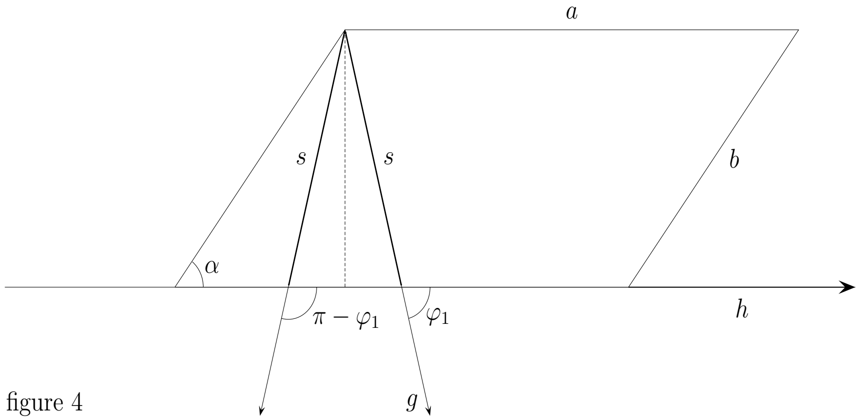

figure 4

If $\varphi \in]\varphi_1, \pi - \varphi_1[$, the test body intersects on g segments of length less than s and thus $x(\varphi) = 0$. If $\varphi \in [0, \varphi_1] \cup [\pi - \varphi_1, \pi - \alpha]$, the distance $x(\varphi)$ can be calculated by utilizing (2) and if $\varphi \in [\pi - \alpha, \pi]$ by utilizing (3). Thus we get

$$\int_0^{\pi} x(\varphi)d\varphi = \int_0^{\varphi_1} x(\varphi)d\varphi + \int_{\pi-\varphi_1}^{\pi-\alpha} x(\varphi)d\varphi + \int_{\pi-\alpha}^{\pi} x(\varphi)d\varphi = ... =$$

$$a\,[1 - \cos(\varphi_1)] + b\,[1 - \cos(\alpha)\cos(\varphi_1)] - s\left[1 - \left(\varphi_1 - \alpha\right)\cot(\alpha) - \cot(\alpha)\sin(\varphi_1)\cos(\varphi_1)\right] =$$
$$a\,[1 - \cos(\varphi_1)] + b - s\left[1 + \left(\varphi_1 - \alpha\right)\cot(\alpha)\right].$$

In this way we have demonstrated the following:

Proposition 2. If $b\sin(\alpha) \le s \le \min\{b, a\sin(\alpha)\}$**, the required probability** p_s **is:**

$$p_s = \frac{2}{\pi A}\Big\{a\,[1 - \cos(\varphi_1)] + b - s\left[1 + \left(\varphi_1 - \alpha\right)\cot(\alpha))\right]\Big\}. \tag{6}$$

Remark: If $b\sin(\alpha) = s$, then $\varphi_1 = \frac{\pi}{2}$ and the formulas (5) and (6) are equal.

The second "not small" case is (B): $b \le s \le a\sin(\alpha)$.

This case is common to the situations (I) and (II). Let $\varphi_1 \in \left[0, \frac{\pi}{2}\right]$ be again uniquely determined by the equality $\sin(\varphi_1) = \frac{b\sin(\alpha)}{s}$, but this time with $\varphi_1 \le \alpha$. As in the case (A) if $\varphi \in]\varphi_1, \pi - \varphi_1[$, we obtain $x(\varphi) = 0$. If $\varphi \in [0, \varphi_1]$, the distance $x(\varphi)$ is given by (2) and if $\varphi \in [\pi - \varphi_1, \pi]$, the distance $x(\varphi)$ is given by (3). Hence

$$\int_0^{\pi} x(\varphi)d\varphi = \int_0^{\varphi_1} x(\varphi)d\varphi + \int_{\pi-\varphi_1}^{\pi} x(\varphi)d\varphi = ... =$$

$$[a + b\cos(\alpha)]\,[1 - \cos(\varphi_1)] + 2b\sin(\frac{\varphi_1}{2})\sin(\alpha - \frac{\varphi_1}{2}) - s\sin^2(\varphi_1) = a\,[1 - \cos(\varphi_1)]$$

and due to the formula (1) we have the following:

Proposition 3. If $b \le s \le a\sin(\alpha)$, the required probability p_s is:

(7)
$$p_s = \frac{2}{\pi A}\Big\{a\,[1-\cos(\varphi_1)]\Big\}.$$

Remark: If $b = s$, then $\varphi_1 = \alpha$ and the formulas (6) and (7) are both equal to $\frac{2a}{\pi A}\,[1-\cos(\alpha)]$.

The third "not small" case is (C): $\max\{b, \sin(\alpha)\} \le s \le \min\{a, d\}$.

This case is common to the situations (I), (II), (III) and (IV). If $\gamma_a \ge \frac{\pi}{2}$, we can calculate p_s by the formula (7). If specially $\gamma_a = \frac{\pi}{2}$, then $d = s = a\sin(\alpha)$ and $\gamma_b = \frac{\pi}{2} - \alpha = \varphi_1$; thus the formula (7) turns to

(7')
$$p_s = p_d = \frac{2}{\pi A}\,\{a\,[1-\sin(\alpha)]\}\,.$$

If $\gamma_a < \frac{\pi}{2}$, we consider φ_1 as in the previous cases and additionally $\varphi_2 \in \left]0, \frac{\pi}{2} - \alpha\right[$ (see fig. 5) determined by $\frac{s}{\sin(\alpha)} = \frac{a}{\sin(\alpha+\varphi_2)}$.

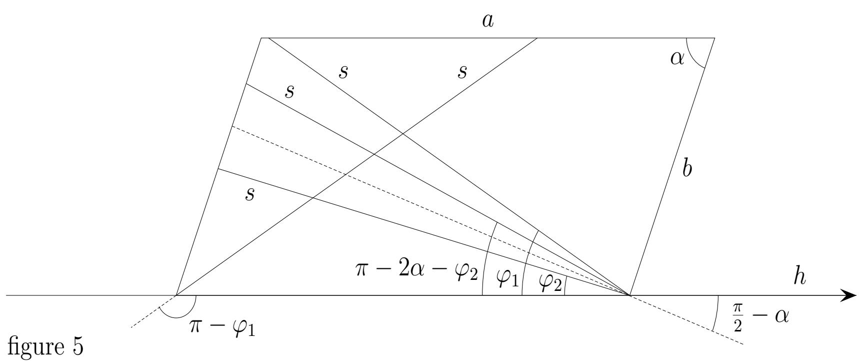

figure 5

If $\varphi \in]\varphi_2, \pi - 2\alpha - \varphi_2[\cup]\varphi_1, \pi - \varphi_1[$, we observe $x(\varphi) = 0$. If $\varphi \in [0, \varphi_2] \cup [\pi - 2\alpha - \varphi_2, \varphi_1]$ and if $\varphi \in [\pi - \varphi_1, \pi]$, by utilizing formula (2) and respectively formula (3), we have

$$\int_0^{\pi} x(\varphi)d\varphi = \int_0^{\varphi_2} x(\varphi)d\varphi + \int_{\pi-2\alpha-\varphi_2}^{\varphi_1} x(\varphi)d\varphi + \int_{\pi-\varphi_1}^{\pi} x(\varphi)d\varphi =$$

$$\left(\int_0^{\varphi_2} + \int_{\pi-2\alpha-\varphi_2}^{\varphi_1}\right)\left[\frac{D}{2}\sin(\beta_b+\varphi) - \frac{s}{2}\cot(\alpha) + \frac{s}{2\sin(\alpha)}\cos(\alpha+2\varphi)\right]d\varphi +$$

$$\int_{\pi-\varphi_1}^{\pi}\left[\frac{D}{2}\sin(\beta_b+\varphi) - b\sin(\alpha+\varphi) + \frac{s}{2}\cot(\alpha) - \frac{s}{s\sin(\alpha)}\cos(\alpha+2\varphi)\right]d\varphi =$$

$$[a+b\cos(\alpha)]\,[1-\cos(\varphi_1)]-D\cos(\beta_a)\cos(\alpha+\varphi_2)+2b\sin(\frac{\varphi_1}{2})\sin(\alpha-\frac{\varphi_1}{2})+$$
$$s\left[(\frac{\pi}{2}-\alpha-\varphi_2)\cot(\alpha)-\sin^2(\varphi_1)\right]+\frac{s}{2}\cot(\alpha)\sin(2\alpha+2\varphi_2)=$$
$$a\,[1-\cos(\varphi_1)]-b\cos(\alpha+\varphi_2)+s\big[\frac{\pi}{2}-(\alpha+\varphi_2)\big]\cot(\alpha).$$

In this way we have demonstrated the following:

Proposition 4. If $\max\{b,\sin(\alpha)\}\le s\le\min\{a,d\}$**, the required probability** p_s **is:**

$$p_s=\frac{2}{\pi A}\left\{a\,[1-\cos(\varphi_1)]-b\cos(\alpha+\varphi_2)+s\big[\frac{\pi}{2}-(\alpha+\varphi_2)\big]\cot(\alpha)\right\}. \tag{8}$$

Remark: If $b\le a\sin(\alpha)=s$, then $\varphi_2=\frac{\pi}{2}-\alpha$ and thus (8) is equal to (7).

The next case is (D): $\max\{b,d,a\sin(\alpha)\}\le s\le a$.

This case is common to the situations (I), (IV) and (V). If φ_1 and φ_2 are defined as before, then $x(\varphi)=0$ if $\varphi\in\,]\varphi_2,\pi-\varphi_1[$.

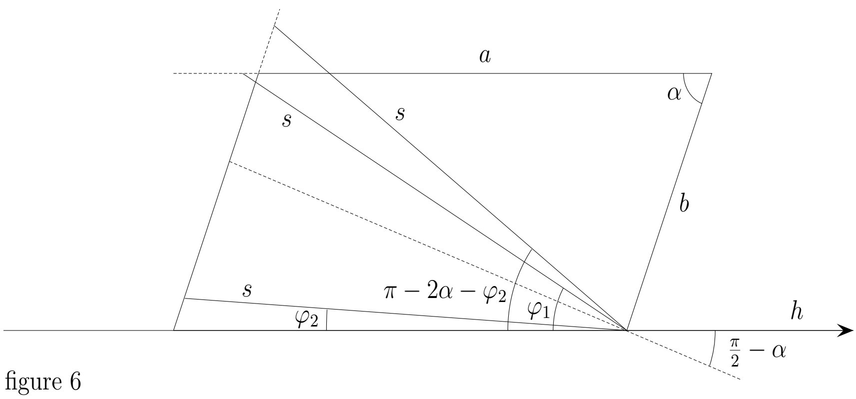

figure 6

If $\varphi\in[0,\varphi_2]$ and if $\varphi\in[\pi-\varphi_1,\pi]$, by utilizing formula (2) and respectively formula (3), we get

$$\int_0^{\pi}x(\varphi)d\varphi=\int_0^{\varphi_2}\left[\frac{D}{2}\sin(\beta_b+\varphi)-\frac{s}{2}\cot(\alpha)+\frac{s}{2\sin(\alpha)}\cos(\alpha+2\varphi)\right]d\varphi+$$
$$\int_{\pi-\varphi_1}^{\pi}\left[\frac{D}{2}\sin(\beta_b+\varphi)-b\sin(\alpha+\varphi)+\frac{s}{2}\cot(\alpha)-\frac{s}{2\sin(\alpha)}\cos(\alpha+2\varphi)\right]d\varphi=$$
$$D\cos(\beta_b)-D\cos(\beta_b+\frac{\varphi_2-\varphi_1}{2})\cos(\frac{\varphi_1+\varphi_2}{2})+2b\sin(\frac{\varphi_1}{2})\sin(\alpha-\frac{\varphi_1}{2})-$$
$$\frac{s}{2}\left[1+(\varphi_2-\varphi_1)\cot(\alpha)-\frac{\sin(\alpha+\varphi_2-\varphi_1)\cos(\varphi_1+\varphi_2)}{\sin(\alpha)}\right].$$

In this way we have demonstrated the following:

Proposition 5. **If** $\max\{b, d, a\sin(\alpha)\} \leq s \leq a$**, the required probability** p_s **is:**

$$
(9) \qquad p_s = \frac{2}{\pi A}\Bigg\{a + b\cos(\alpha) - D\cos(\beta_b + \frac{\varphi_2 - \varphi_1}{2})\cos(\frac{\varphi_1 + \varphi_2}{2}) + 2b\sin(\frac{\varphi_1}{2})\sin(\alpha - \frac{\varphi_1}{2}) - \frac{s}{2}\left[1 + (\varphi_2 - \varphi_1)\cot(\alpha) - \frac{\sin(\alpha + \varphi_2 - \varphi_1)\cos(\varphi_1 + \varphi_2)}{\sin(\alpha)}\right]\Bigg\}.
$$

Remark: If $b \leq a\sin(\alpha) = d = s$, then $\varphi_1 = \varphi_2 = \frac{\pi}{2} - \alpha$ and thus the formulas (8) and (9) are both equal to $p_s = p_{a\sin(\alpha)} = p_d = \frac{2}{\pi A}\left\{a\left[1 - sin(\alpha) - \sin(\alpha)\cos^2(\alpha)\right] + b\sin(\alpha)\cos(\alpha)\right\}$.

The next case is (E): $a \leq s \leq d$.

This case is common to the situations (II) and (III). We note that (see figure 7) $x(\varphi) = 0$ if we have $\varphi \in]0, \pi - 2\alpha - \varphi_3[\cup]\varphi_1, \pi - \varphi_1[\cup]\pi - \varphi_3, \pi[$. ($\varphi_3$ is determined by $\frac{s}{\sin\alpha} = \frac{a}{\sin(\alpha - \varphi_3)}$.)

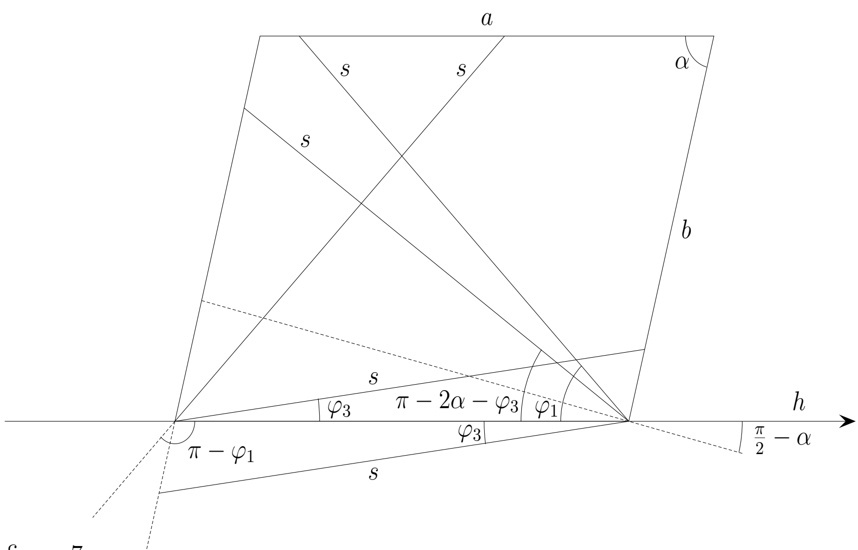

figure 7

If $\varphi \in [\pi - 2\alpha - \varphi_3, \varphi_1]$ and if $\varphi \in [\pi - \varphi_1, \pi - \varphi_3]$, by utilizing formulas (2) and (3) for the calculation of $x(\varphi)$, we have

$$
\int_0^{\pi} x(\varphi)d\varphi = \int_{\pi - 2\alpha - \varphi_3}^{\varphi_1} \left[\frac{D}{2}\sin(\beta_b + \varphi) - \frac{s}{2}\cot(\alpha) + \frac{s}{2\sin(\alpha)}\cos(\alpha + 2\varphi)\right]d\varphi +
$$

$$
\int_{\pi - \varphi_1}^{\pi - \varphi_3} \left[\frac{D}{2}\sin(\beta_b + \varphi) - b\sin(\alpha + \varphi) + \frac{s}{2}\cot(\alpha) - \frac{s}{2\sin(\alpha)}\cos(\alpha + 2\varphi)\right]d\varphi =
$$

$$
D\sin(\alpha + \varphi_3 - \beta_b)\sin(\alpha) - [a + b\cos(\alpha)]\cos(\varphi_1) + 2b\sin(\frac{\varphi_1 - \varphi_3}{2})\sin(\alpha - \frac{\varphi_1 + \varphi_3}{2}) + s[\frac{\pi}{2} - \alpha - \varphi_3]\cot(\alpha) + \frac{s}{2}\cos(2\varphi_1) + \frac{s}{2\sin(\alpha)}\sin(\alpha + 2\varphi_3)\cos(2\alpha).
$$

In this way we have demonstrated the following:

Proposition 6. If $a \le s \le d$, **the required probability** p_s **is:**

$$p_s = \frac{2}{\pi A}\Big\{D\sin(\alpha+\varphi_3-\beta_b)\sin(\alpha) - [a+b\cos(\alpha)]\cos(\varphi_1) + 2b\sin(\frac{\varphi_1-\varphi_3}{2})\sin(\alpha-\frac{\varphi_1+\varphi_3}{2}) +$$

$$s\big[\frac{\pi}{2}-\alpha-\varphi_3\big]\cot(\alpha) + \frac{s}{2}\cos(2\varphi_1) + \frac{s}{2\sin(\alpha)}\sin(\alpha+2\varphi_3)\cos(2\alpha)\Big\}. \tag{10}$$

Remark: If $d = s = a$, then $\varphi_2 = \varphi_3 = 0$, $\varphi_1 = \pi - 2\alpha$ and $b = 2a\cos(\alpha)$; thus the formulas (9) and (10) are both equal to $p_d = p_s = p_a = \frac{2}{\pi A}\left\{\left(\frac{\pi}{2}-\alpha\right)\cot(\alpha)\right\}$.

The next case is (F): $a\sin(\alpha) \le s \le \min\{b, d\}$.

This case is common to the situations (III), (IV) and (V). If $\gamma_a \ge \frac{\pi}{2}$, then p_s can be calculated by means of formula (6). If $\gamma_a < \frac{\pi}{2}$, then utilizing the previous notations, we have $\pi - \varphi_1 \le \pi - \alpha$ (see fig. 8) and thus $x(\varphi) = 0$ if $\varphi \in]\varphi_2, \pi - 2\alpha - \varphi_2[\cup]\varphi_1, \pi - \varphi_1[$.

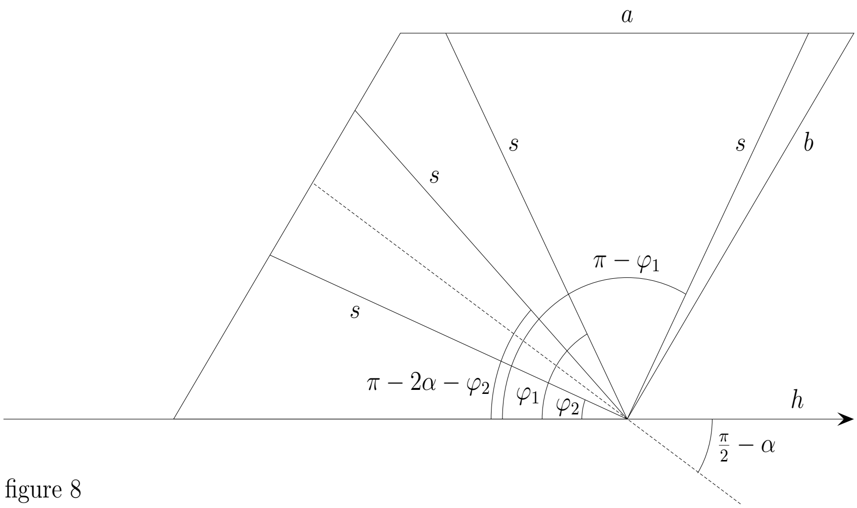

figure 8

Hence

$$\int_0^{\pi} x(\varphi)d\varphi = \left(\int_0^{\varphi_2} + \int_{\pi-2\alpha-\varphi_2}^{\varphi_1} + \int_{\pi-\varphi_1}^{\pi-\alpha}\right)\left[\frac{D}{2}\sin(\beta_b+\varphi) - \frac{s}{2}\cot(\alpha) + \frac{s}{2\sin(\alpha)}\cos(\alpha+2\varphi)\right]d\varphi +$$

$$\int_{\pi-\alpha}^{\pi}\left[\frac{D}{2}\sin(\beta_b+\varphi) - b\sin(\alpha+\varphi) + \frac{s}{2}\cot(\alpha) - \frac{s}{2\sin(\alpha)}\cos(\alpha+2\varphi)\right]d\varphi =$$

$$D\cos(\beta_b)\,[1-\cos(\varphi_1)] - D\cos(\beta_a)\cos(\alpha+\varphi_2) + b\,[1-\cos(\alpha)] +$$

$$s\big[\frac{\pi}{2}-\varphi_1-\varphi_2\big]\cot(\alpha) - s + s\cot(\alpha)\sin(\alpha+\varphi_1+\varphi_2)\cos(\alpha+\varphi_2-\varphi_1) =$$

$$a\,[1-\cos(\varphi_1)] + b\,[1-\cos(\alpha+\varphi_2)] - s\big[1 + \big(\varphi_1+\varphi_2-\frac{\pi}{2}\big)\cot(\alpha)\big].$$

In this way we have demonstrated the following:

Proposition 7. Let be $a\sin(\alpha) \le s \le \min\{b, d\}$. In this case the required probability p_s is given by (6) if $\gamma_a \ge \frac{\pi}{2}$ and if $\gamma_a < \frac{\pi}{2}$ we have

$$p_s = \frac{2}{\pi A}\Big\{a\,[1-\cos(\varphi_1)] + b\,[1-\cos(\alpha+\varphi_2)] - s\big[1+\big(\varphi_1+\varphi_2-\frac{\pi}{2}\big)\cot(\alpha)\big]\Big\}. \tag{11}$$

Remark: If $a\sin(\alpha) = s \le b \le d$, we obtain $\varphi_2 = \frac{\pi}{2} - \alpha$; thus the formulas (6) and (11) are both equal to $p_{a\sin(\alpha)} = \frac{2}{\pi A}\{a\,[1-\cos(\varphi_1) - \sin(\alpha) - (\varphi_1-\alpha)\cos(\alpha)] + b\}$.

The next case is (G): $d \le s \le b$.

This case appears only in the situation (V). We note that $x(\varphi) = 0$ if $\varphi \in]\varphi_2, \pi-\varphi_1[$ (see fig.9).

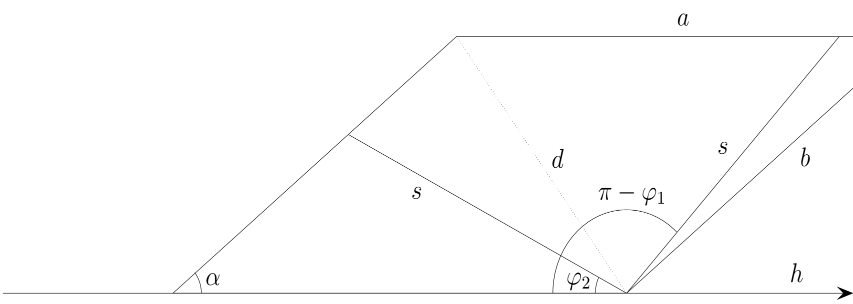

figure 9

Thus

$$\int_0^{\pi} x(\varphi)d\varphi = \left(\int_0^{\varphi_2} + \int_{\pi-\varphi_1}^{\pi-\alpha}\right)\left[\frac{D}{2}\sin(\beta_b+\varphi) - \frac{s}{2}\cot(\alpha) + \frac{s}{2\sin(\alpha)}\cos(\alpha+2\varphi)\right]d\varphi +$$
$$\int_{\pi-\alpha}^{\pi}\left[\frac{D}{2}\sin(\beta_b+\varphi) - b\sin(\alpha+\varphi) + \frac{s}{2}\cot(\alpha) - \frac{s}{2\sin(\alpha)}\cos(\alpha+2\varphi)\right]d\varphi =$$
$$D\cos(\beta_b) - \frac{D}{2}\cos(\beta_b+\varphi_2) - \frac{D}{2}\cos(\beta_b-\varphi_1) + b\,[1-\cos(\alpha)] +$$
$$s(\alpha - \frac{\varphi_1+\varphi_2}{2})\cot(\alpha) - s + \frac{s}{2\sin(\alpha)}\sin(\varphi_1+\varphi_2)\cos(\alpha+\varphi_2-\varphi_1) =$$
$$a\big[1 - \frac{\cos(\varphi_1)}{2}\big] + b\big[1 - \frac{1}{2}\cos(\alpha+\varphi_2)\big] + s\big[(\alpha - \frac{\varphi_1+\varphi_2}{2})\cot(\alpha) - \frac{3}{2}\big].$$

In this way we have demonstrated the following:

Proposition 8. If $d \leq s \leq b$, the required probability p_s is:

$$p_s = \frac{2}{\pi A}\left\{a\big[1-\frac{\cos(\varphi_1)}{2}\big] + b\big[1-\frac{1}{2}\cos(\alpha+\varphi_2)\big] + s\big[(\alpha-\frac{\varphi_1+\varphi_2}{2})\cot(\alpha)-\frac{3}{2}\big]\right\}. \tag{12}$$

Remark: For $d = s \leq b$ we have:

- If $\gamma_a < \frac{\pi}{2}$, then $\varphi_2 < \varphi_1 = \pi - 2\alpha - \varphi_2$ and the formulas (11) and (12) are both equal to

$$\begin{aligned} p_s &= \frac{2}{\pi A}\Big\{a\big[1-\cos(\varphi_1)\big] + b\big[1+\cos(\alpha+\varphi_1)\big] - d\big[1+\big(\frac{\pi}{2}-2\alpha\big)\cot(\alpha)\big]\Big\} \\ &= \frac{2}{\pi A}\Big\{a\big[1-\frac{\cos(\varphi_1)}{2}\big] + b\big[1+\frac{1}{2}\cos(\alpha+\varphi_1)\big] - d\big(\frac{\pi}{2}-2\alpha\big)\cot(\alpha) - \frac{3}{2}d\Big\}. \end{aligned}$$

- If $\gamma_a \geq \frac{\pi}{2}$, then $\varphi_1 = \varphi_2$ and the formulas (11) and (12) are both equal to

$$\begin{aligned} p_d &= \frac{2}{\pi A}\Big\{a\big[1-\cos(\varphi_1)\big] + b - d\big[1+(\varphi_1-\alpha)\cot(\alpha)\big]\Big\} \\ &= \frac{2}{\pi A}\Big\{a\big[1-\frac{\cos(\varphi_1)}{2}\big] + b\big[1-\frac{1}{2}\cos(\alpha+\varphi_1)\big] + d\big[(\alpha-\varphi_1)\cot(\alpha)-\frac{3}{2}\big]\Big\}. \end{aligned}$$

The last case is (H): $\max\{a, d\} \leq s \leq D$.

This case is common to all situations (I), (II), (III), (IV) and (V). Now we note that $x(\varphi) = 0$ except if $\varphi \in [\pi-\varphi_1, \pi-\varphi_3]$ (see fig.10). Since $b \leq a \leq s$, we observe $\varphi_1 < \alpha$ and $[\pi-\varphi_1, \pi-\varphi_3] \subset [\pi-\alpha, \pi]$.

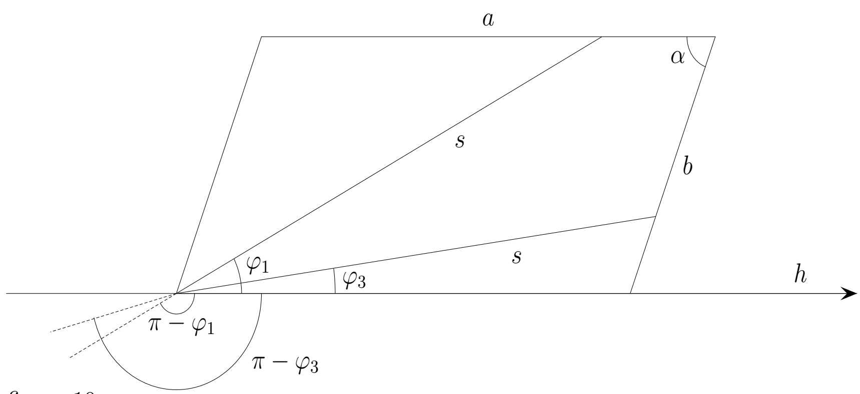

figure 10

Thus

$$\int_0^{\pi} x(\varphi)d\varphi = \int_{\pi-\varphi_1}^{\pi-\varphi_3}\left[\frac{D}{2}\sin(\beta_b+\varphi) - b\sin(\alpha+\varphi) + \frac{s}{2}\cot(\alpha) - \frac{s}{2\sin(\alpha)}\cos(\alpha+2\varphi)\right]d\varphi =$$

$$\begin{gathered}\frac{D}{2}\left[\cos(\beta_b-\varphi_3)-\cos(\beta_b-\varphi_1)\right]+b\left[\cos(\alpha-\varphi_1)-\cos(\alpha-\varphi_3)\right]+\\ \frac{s}{2}(\varphi_1-\varphi_3)\cot(\alpha)-\frac{s}{4\sin(\alpha)}\left[\sin(\alpha-2\varphi_3)-\sin(\alpha-2\varphi_1)\right]=\\ D\sin(\frac{\varphi_3-\varphi_1}{2})\sin(\beta_b-\frac{\varphi_1+\varphi_3}{2})+2b\sin(\frac{\varphi_3-\varphi_1}{2})\sin(\alpha-\frac{\varphi_1+\varphi_3}{2})+\\ \frac{s}{2}(\varphi_1-\varphi_3)\cot(\alpha)-\frac{s}{2\sin(\alpha)}\sin(\varphi_1-\varphi_3)\cos(\alpha-\varphi_1-\varphi_3).\end{gathered}$$

In this way we have demonstrated the following:

Proposition 9. If $\max\{a,d\}\le s\le D$**, the required probability** p_s **is:**

$$p_s=\frac{2}{\pi A}\Big\{D\sin(\frac{\varphi_3-\varphi_1}{2})\sin(\beta_b-\frac{\varphi_1+\varphi_3}{2})+2b\sin(\frac{\varphi_1-\varphi_3}{2})\sin(\alpha-\frac{\varphi_1+\varphi_3}{2})+ \\ \frac{s}{2}(\varphi_1-\varphi_3)\cot(\alpha)-\frac{s}{2\sin(\alpha)}\sin(\varphi_1-\varphi_3)\cos(\alpha-\varphi_1-\varphi_3)\Big\}. \tag{13}$$

Remarks:

1) If $d\le a=s$, then $\varphi_3=0$ and by (13) and (9) we have

$$p_s=\frac{2}{\pi A}\Big\{D\sin(\frac{\varphi_1}{2})\sin(\frac{\varphi_1}{2}-\beta_b)+2b\sin(\frac{\varphi_1}{2})\sin(\alpha-\frac{\varphi_1}{2})+ \\ \frac{s}{2}\varphi_1\cot(\alpha)-\frac{s}{2\sin(\alpha)}\sin(\varphi_1)\cos(\alpha-\varphi_1)\Big\}$$

respectively

$$p_s=\frac{2}{\pi A}\Big\{D\cos(\beta_b)-D\cos(\beta_b-\frac{\varphi_1}{2})\cos(\frac{\varphi_1}{2})+2b\sin(\frac{\varphi_1}{2})\sin(\alpha-\frac{\varphi_1}{2})- \\ \frac{s}{2}+\frac{s}{2}\varphi_1\cot(\alpha)+\frac{s}{2}\frac{\sin(\alpha-\varphi_1)\cos(\varphi_1)}{\sin(\alpha)}\Big\}.$$

The previous expressions are equal because: $\cos(\beta_b)=\cos(\beta_b-\frac{\varphi_1}{2})\cos(\frac{\varphi_1}{2})-\sin(\beta_b-\frac{\varphi_1}{2})\sin(\frac{\varphi_1}{2})$ and $\sin(\alpha)=\sin(\varphi_1)\cos(\alpha-\varphi_1)+\sin(\alpha-\varphi_1)\cos(\varphi_1)$.

2) If $s=D$, then $\varphi_1=\varphi_2=\beta_b$ and thus, by (13), $p_D=0$ (this is evident also for geometric considerations).

Final remark: The results, we derived in this article, can also be considered as calculations of the chord-length distribution for the parallelogram P. Namely let $F:[0,D]\to[0,1]$ be the the chord-length distribution and $\overline{F}:=1-F$, then maintaining our notations we obtain

$$\overline{F}(s)\ =\ p_s\,/\,p_0\ =\ \frac{2\int\limits_0^{\pi}x(\varphi)d\varphi}{\pi A}\ /\ \frac{2(a+b)}{\pi A}\ =\ \frac{1}{a+b}\int\limits_0^{\pi}x(\varphi)d\varphi.$$

So the density of distribution is $f = \frac{\partial F}{\partial s} = -\frac{\partial \overline{F}}{\partial s}$. For a "small" parallelogram Proposition 1 shows

$$f(s) \ = \ 1 + \left(\frac{\pi}{2} - \alpha\right)\cot(\alpha),$$

i.e. f is constant. In the other case (,thus P is not "small",) the result is more complicated, because p_s depends not only explicitly on s but also implicitly, because the angles φ_1, φ_2 and φ_3 depend on s as well.

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PROBABILITÀ GEOMETRICHE PER RETICOLI ESAGONALI CON OSTACOLI ESAGONALI

ANDREI DUMA – MARIUS I. STOKA

Si risolvono problemi di tipo Buffon per reticoli esagonali con ostacoli esagonali. I corpi test considerati sono rettangoli di lati costanti, triangoli equilateri di lato costante ed esagoni regolari di lato costante.

In due lavori precedenti, [1], [2], abbiamo studiato problemi di probabilità geometriche di tipo Buffon per reticoli quadratici con ostacoli quadratici e per reticoli triangolari con ostacoli triangolari. I corpi test considerati sono stati segmenti, rettangoli, circoli e triangoli. In questo lavoro facciamo uno studio analogo per reticoli esagonali con ostacoli esagonali.

Sia, nel piano euclideo E_2 riferito ad un sistema di assi ortogonali Q_x, O_y un reticolo (virtuale) $\Re(A, a)$ la cui cellula fondamentale è un esagono regolare ξ (con i lati tratteggiati) di lato A e tale che in ogni suo vertice si trovi un esagono regolare (ostacolo) di lato $a < A/2$, (fig. 1).

Nella figura 2 abbiamo indicato, a sinistra, la cellula ξ e, a destra, il suo completamento con i pezzi degli ostacoli contenuti in ξ.

Sia poi T un corpo test che può essere un rettangolo R di lati costanti l e k con la diagonale $d = \sqrt{l^2 + k^2}$ tale che $d < A - 2a$, o un triangolo equilatere Δ di lato costante $l < A - 2a$, o un esagono regolare E di lato costante $l < \dfrac{A}{2} - a$.

AMS 2000 Subject Classification: Geometric Probability, stokastic geometry, random sets, random convex sets and integral geometry.
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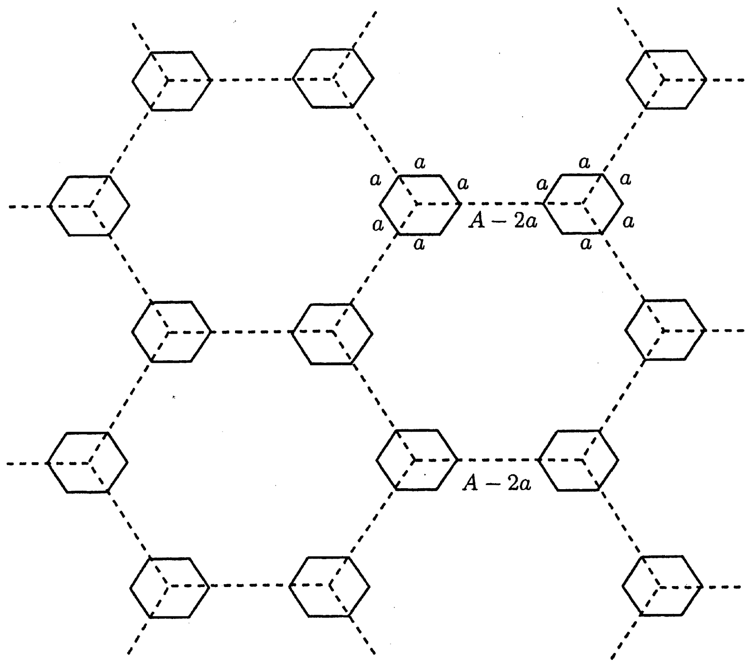

Fig. 1

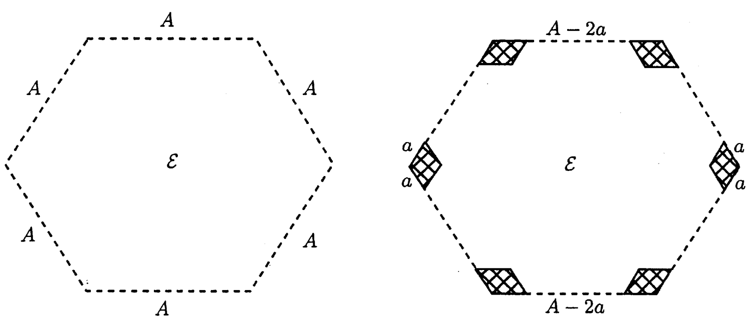

Fig. 2

In quello che segue, vogliamo calcolare la propabilità $P_{a,A,T}$ che il corpo test T uniformemente distribuito in una regione limitata del piano non intersechi gli ostacoli.

Denotiamo con P il baricentro del corpo test T e con v una direzione fissa intrinsicamente legata a T.

Indicando con $\mathcal{M}$ l'insieme dei corpi test T che hanno il baricentro P in ξ e con $\mathcal{N}$ l'insieme dei corpi test T con i baricentri situati in ξ ma che non intersecano gli ostacoli associati ad ξ, abbiano [1, pag. 20]

$$P_{a,A,T} = \frac{\mu(\mathcal{N})}{\mu(\mathcal{M})}, \tag{1}$$

dove μ è la misura di Lebesgue.

Per calcolare le misure $\mu(\mathcal{M})$ e $\mu(\mathcal{N})$ si usa la misura cinematica di Poincaré [3, pag. 126]

$$dk = dx \wedge dy \wedge d\varphi,$$

dove x, y sono le coordinate del baricentro P e φ è l'angolo tra l'asse O_x e la direzione v.

1. Consideriamo daprima il caso in cui il corpo test si un rettangolo R di lati costanti l e k, con la diagonale $d = \sqrt{l^2 + k^2} < A - 2a$; inoltre sia δ l'angolo tra il lato l e la diagonale d, cioè $d \cos\delta = l$, $d \sin\delta = k$.

THEOREM 1. *Se $d = \sqrt{l^2 + k^2} < A - 2a$ la probabilità che un rettangolo R, di lati costanti l e k, uniformemente distribuito in una regione limitata del piano, non intersechi gli ostacoli del reticolo $\mathcal{R}(A, a)$ è*

$$P_{a,A,R} = 1 - \frac{2a^2}{A^2} - \frac{4lk}{3\sqrt{3}A^2} - \frac{8a(l+k)}{\sqrt{3}\pi A^2}. \tag{2}$$

Dimonstrazione. Per ragione di simmetria è sufficiente che l'angolo φ varia soltanto tra 0 e $\pi/6$. Allora

$$\mu(\mathcal{M}) = \int_0^{\pi/6} d\varphi \iint_{\{(x,y)\in\xi\}} dx\,dy = \int_0^{\pi/6} (\text{area}\,\xi)\,d\varphi = \frac{\pi}{6} \cdot \frac{3\sqrt{3}A^2}{2} = \frac{\sqrt{3}\pi A^2}{4}. \tag{3}$$

Per calcolare la misura $\mu(\mathcal{N})$, definiamo il poligono $\xi(\varphi)$, l'insieme dei punti $P \in \xi$ tale che il rettangolo R, di baricentro P e l'angolo corrispondente φ, non intersechi gli ostacoli. I poligoni $\xi(\varphi)$ hanno diverse forme, per esempio come nella figura 3.

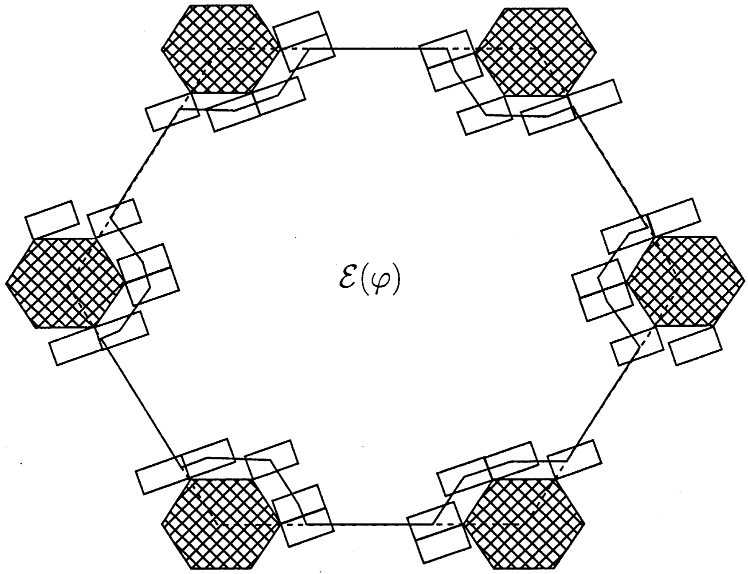

Fig. 3

Ma, in tutti i casi, area $\xi(\varphi)$ è sempre la stessa area

$$\xi(\varphi) = \frac{3\sqrt{3}A^2}{2} - 2\left\{\frac{3\sqrt{3}a^2}{2} + lk + 2\cdot\right.$$
$$\left.\cdot\frac{ad}{2}\left[\sin(\varphi+\delta) + \sin\left(\frac{2\pi}{3} - \varphi - \delta\right) + \sin\left(\frac{\pi}{3} - \varphi + \delta\right)\right]\right\}.$$

Pertanto

$$\mu(\mathcal{N}) = \int_0^{\pi/6} d\varphi \iint_{\{(x,y)\in\xi(\varphi)\}} dx\,dy = \int_0^{\pi/6} [\text{area}\,\xi(\varphi)]d\varphi$$
$$= \frac{\pi}{6}\left(\frac{3\sqrt{3}A^2}{2} - 3\sqrt{3}a^2 - 2lk\right) - 2ad(\sin\delta + \cos\delta) \tag{4}$$
$$= \frac{\pi}{6}\left(\frac{3\sqrt{3}A^2}{2} - 3\sqrt{3}a^2 - 2lk\right) - 2a(l+k).$$

Le formule (1), (3), (4) ci danno la probabilità (2). □

Ossservazioni. 1° Se $k = 0$, il rettangolo R diventa un segmento S di lunghezza costante $l < A - 2a$ e la probabilità corrispondente $P_{a,A,s}$ si scrive

$$P_{a,A,s} = 1 - \frac{2a^2}{A^2} - \frac{8al}{\sqrt{3}\pi A^2}.$$

2° Se $k = l$, il rettangolo R diventa un quadrato Q di lato costante $l < (A - 2a)/\sqrt{2}$ e la probailità corrispondente $P_{a,A,Q}$ si scrive

$$P_{a,A,Q} = 1 - \frac{2a^2}{A^2} - \frac{4l^2}{3\sqrt{3}A^2} - \frac{16al}{\sqrt{3}\pi A^2}.$$

3° Il risultato non cambia se ogni ostacolo esagonale ha una posizione arbitraria che si ottiene mediante una rotazione arbitraria di ogni ostacolo attorno al suo baricentro che resta fisso (i vertici della cellula fondamentale ξ). La restrizione $d < A - 2a$ resta valida in quanto in questo caso il rettangolo R è piccolo rispetto al reticolo $\mathcal{R}(A, a)$ (cioè la distanza tra ogni due ostacoli è più grande della diagonale d di R).

2. Consideriamo ora il caso in cui il corpo test sia un triangolo equilatere Δ di lato $l < A - 2a$.

TEOREMA 2. *Se $l < A - 2a$, la probabilità che un triangolo equilatere Δ di lato constante l, uniformemente distribuito in una regione limitata del piano non intersechi gli ostacoli del reticolo $\mathcal{R}(A, a)$ è*

$$P_{a,A,\Delta} = 1 - \frac{2a^2}{A^2} - \frac{4\sqrt{3}al}{\pi A^2} - \frac{l^2}{3A^2}. \tag{5}$$

Dimostrazione. Come nel caso precedente, il poligono $\xi(\varphi)$ è l'insieme dei punti $P \in \xi$ tale che il triangolo Δ di baricentro P e l'angolo corrispondente φ, non intersechi gli ostacoli.

Anche in questo caso i poligoni $\xi(\varphi)$ hanno diverse forme, per esempio come nella figura 4.

L'area di $\xi(\varphi)$ è sempre la stessa

$$\text{area } \xi(\varphi)\frac{3\sqrt{3}A^2}{2} - 2\left\{\frac{3\sqrt{3}a^2}{2} + \frac{\sqrt{3}l^2}{4} + \sqrt{3}al\left[\cos\varphi + \sin\left(\frac{\pi}{3} + \varphi\right)\right]\right\}.$$

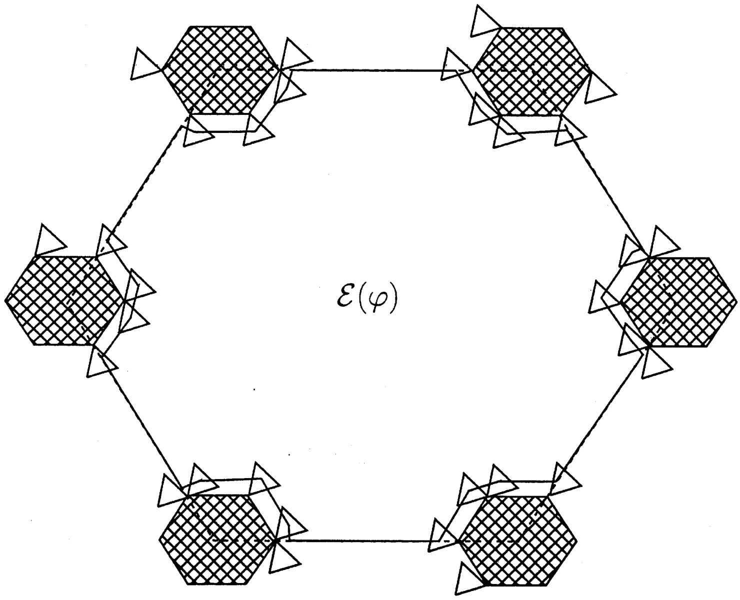

Fig. 4

In questo caso l'angolo φ varia tra O e $\pi/3$, quindi

$$(3') \qquad \mu(\mathcal{M}) = \int_0^{\pi/3} d\varphi \iint_{\{(x,y)\in\xi\}} dxdy = \frac{\sqrt{3}\pi A^2}{2}$$

e

$$(4') \quad \mu(\mathcal{N}) = \int_0^{\pi/3} [\text{area}\,\xi(\varphi)]dy = \frac{\pi}{3}\left(\frac{3\sqrt{3}A^2}{2} - 3\sqrt{3}a^2 - \frac{\sqrt{3}l^2}{2}\right) - 6al.$$

Le formule (1), (3') e (4') ci danno la probabilità (5) □

Osservazione. Questo risultato non cambia se ogni ostacolo esagonale subisce una rotazione arbitraria attorno al suo baricentro.

3. Infine, consideriamo come corpo test un esagono regolare E di lato $l < \dfrac{A}{2} - a$.

TEOREMA 3. *Se* $l < \dfrac{A}{2} - a$, *la probabilità che un esagono regolare* E *di lato costante* l, *uniformemente distribuito in una regione limitata del* *piano non intersechi gli ostacoli del reticolo $\mathcal{R}(A, a)$ è*

$$p = 1 - \frac{2a^2}{A^2} - \frac{8\sqrt{3}al}{\pi A^2} - \frac{2l^2}{A^2}. \tag{6}$$

Dimonstrazione. In questo caso abbiamo

$$\operatorname{area} \xi(\varphi) = \frac{3\sqrt{3}}{2}(A^2 - 2a^2 - 2l^2) - 12al \sin\left(\frac{\pi}{3} + \varphi\right),$$

quindi

$$\begin{aligned} \mu(\mathcal{N}) &= \int_0^{\pi/6} [\operatorname{area} \xi(\varphi)] d\varphi \\ &= \int_0^{\pi/6} \left[\frac{3\sqrt{3}}{2}(A^2 - 2a^2 - 2l^2) - 12al \sin\left(\frac{\pi}{3} + \varphi\right) \right] d\varphi \\ &= \frac{3\sqrt{3}}{2}(A^2 - 2a^2 - 2l^2)\frac{\pi}{6} - 6al. \end{aligned} \tag{7}$$

Le formule (1), (3) e (7) ci danno la probabilità (6). □

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MOVES FOR SEGRE PRODUCTS OF VERONESE SUBRINGS

GIOIA FAILLA

ABSTRACT. We consider the Segre product of two square-free Veronese varieties and we determine the set of moves associated to the corresponding semigroup homomorphism. We define the initial vector of a generator movie of the kernel of a semigroup homomorphism by the corresponding initial monomial of the generator binomial of the toric ideal of the lifted algebras homomorphism.

INTRODUCTION

In [2] the role of the semigroup homomorphism π associated to the Segre product of two square-free t-Veronese subrings of two polynomial rings in n and m variables, respectively, has been established, for $t = 2$. If we go on for $t > 2$, the role is very complicated, hence the research of elements of the fibers of π is not easy. Moreover it is always possible to determine the moves of π, because they come from the toric ideal I of the Segre product of the two square free t-Veronese subrings, the kernel of the ring homomorphism that lifts π to a $K-$algebras homomorphism, K a field. Then, by the interplay between the generator binomials of the toric ideal I and the generators of the kernel of π, we obtain the moves of π In this paper, we work in a general context of the Segre product of two square free s-Veronese subring and t-Veronese subrings, $s, t > 2$, of two polynomial rings in n and m variables, respectively.
In N.1 we determine the toric ideal and, by going down,we compute the moves of the semigroup homomorphism associated.
In N.2 we give the definition of initial vector of a movie of a semigroups homomorphism $\pi' : N^n \to Z^m$ in terms of its decomposition as a difference of two positive integer vectors of N^n. We study special moves that come from a Groebner basis of the toric ideal of the $K-$algebras homomorphism that lifts π'. In this way, we define the initial vector of a movie as the vector associated to the initial monomial of a binomial that generates I. As an example,we study the set of moves that minimally generates the kernel of the homomorphism associated to the Segre product of two polynomial rings and we prove that they constitute a Groebner basis for the kernel.

The initial vectors of this set are characterized as $n \times m$ special matrices with only two entries different from 0 and equal to 1.

1. SEGRE PRODUCTS OF TWO VERONESE SUBRINGS

Let n and d be two integer positive numbers. Let $\mathcal{A} = \{\underline{a}_1, ..., \underline{a}_n\}$ a fixed subset of lattice vectors in Z^d. We consider the homomorphism of semigroups:

$$\pi : N^n \longrightarrow Z^d$$

$$\underline{u} \longrightarrow u_1\underline{a}_1 + ... + u_n\underline{a}_n$$

where $\underline{u} = (u_1, ..., u_n)$, and the image of π is the semigroup generated by $\mathcal{A}$, $N\mathcal{A} = \{\lambda_1\underline{a}_1 + ... + \lambda_n\underline{a}_n, \lambda_1, ..., \lambda_n \in N\}$. If K is a field, the map π can be lifted to the K-algebras homomorphism:

$$\hat{\pi} : K[T_1, ..., T_n] \to K[t_1, ..., t_d, t_1^{-1}, ..., t_d^{-1}] = K[\underline{t}^{\pm 1}]$$

$$T_i \to \underline{t}^{\underline{a}_i} = t_1^{a_{i1}} t_2^{a_{i2}} ... t_d^{a_{id}}$$

$$i = 1, ..., n \qquad \underline{a}_i = (a_{i1}, a_{i2}, ..., a_{id}).$$

$\operatorname{Ker}\hat{\pi}$ is denoted by $I_{\mathcal{A}}$ and it's called toric ideal of $\mathcal{A}$. Since $\operatorname{Im}\hat{\pi} \subset K[t^{\pm 1}]$, $I_{\mathcal{A}}$ is a prime ideal and $K[T_1, ..., T_n]/I_{\mathcal{A}} \cong K[\underline{t}^{\underline{a}_1}, ..., \underline{t}^{\underline{a}_n}]$. The affine variety $V(I_{\mathcal{A}})$ in K^n is called *affine toric variety.*

For the following, the semigroup N^n has to be considered as a semigroup of Z^n, otherwise we couldn't speak of $\operatorname{Ker}\pi$.

Definition 1.1. *Let $\underline{u} \in Z^n$. It can be written uniquely $\underline{u} = \underline{u}^+ - \underline{u}^-$, where $\underline{u}^+$ and $\underline{u}^-$ are vectors with non-negative components and disjoint support, where* $\operatorname{supp}\underline{u} = \{i \in N, u_i \neq 0\}$.

Definition 1.2. *An element of* $\operatorname{Ker}\pi \subset Z^n$ *will be called movie of* π.

In [2] and [3] there is a computation of moves of special semigroups homomorphisms, linked to Segre products of polynomial rings of $2-$Veronese square-free monomial subrings. In the general case, let R and P be homogeneous $K-$algebras.

We recall:

Definition 1.3. 1) *The Segre Product $R *_S P$ of R and P is the homogeneous subalgebra of $R \otimes_K P$ with homogeneous components $(R *_s P)_i = R_i \otimes_K P_i$.*
2) *The d-Veronese ring of R is the subring $R_{(d)}$ of R with homogeneous*

component $(R_{(d)})_i = R_{id}$.

Now we consider, for s, t positive integers, the square-free s-Veronese and $t-$Veronese rings $R^{(s)}$, $S^{(t)}$, subrings of $R = K[X_1, ..., X_n]$ and of $P = K[Y_1, ..., Y_m]$ respectively, generated by all square-free monomials of degree s and t in the variables $X_1, ..., X_n$ and $Y_1, ..., Y_m$, that is $R^{(s)} = K[X_1X_2 \dots X_s, \dots, X_{n-s+1} ... X_{n-1}X_n]$ and $S^{(t)} = K[Y_1Y_2 \dots Y_t, ..., Y_{m-t+1} \dots Y_{m-1}Y_m]$.

Consider the Segre product $R^{(s)} *_s S^{(t)} = K[X_1X_2 \dots X_sY_1Y_2 \dots Y_t, ..., X_{n-s+1} ... X_{n-1}X_nY_{m-1}Y_{m-t+1} \dots Y_{m-1}Y_m]$. Let I be the toric ideal of $R^{(s)}$ and J the toric ideal of $S^{(t)}$. Let $U^{(s)} \subset \mathbb{P}_K^{\binom{n}{s}-1}$ be the toric algebraic variety defined by I and $W^{(t)} \subset \mathbb{P}_K^{\binom{m}{t}-1}$ the toric algebraic variety defined by J. Then the algebraic variety defined by the toric ideal of $R^{(s)} *_s S^{(t)}$ is contained in $\mathbb{P}^{N-1}$, where $N = \binom{n}{s}\binom{m}{t} - 1$.

Definition 1.4. *We call* $U^{(s)} *_s W^{(t)} \hookrightarrow \mathbb{P}^{N-1}$ *the Segre immersion in* $\mathbb{P}^{N-1}$ *of the toric varieties* $U^{(s)}$ *and* $W^{(t)}$.

Definition 1.5. *We call fiber of the immersion* $U^{(s)} *_s W^{(t)} \hookrightarrow \mathbb{P}^{N-1}$ *a fiber of the* $K-$*algebras homomorphism* $\hat{\pi}_2 : K[U_1, ..., U_{N-1}] \rightarrow$
$K[X_1X_2 \dots X_sY_1Y_2 \dots Y_t, ..., X_{n-s+1} ... X_{n-1}X_nY_{m-1}Y_{m-t+1} \dots Y_{m-1}Y_m]$.

Theorem 1.6. $\hat{\pi}_2$ *is the lifting of the homomorphism of semigroups rings*

$$\pi_2 : N^{\binom{n}{s}\times\binom{m}{t}} \rightarrow N^n \oplus N^m.$$

for the configuration $\mathcal{A}_2 = \{e_{i_1} + e_{i_2} + \dots + e_{i_s} \oplus e'_{j_1} + e'_{j_2} + \dots + e'_{j_t}, 1 \leq i_1 < i_2 < \dots < i_s \leq n, 1 \leq j_1 < j_2 < \dots j_s \leq m\} \subset N^n \oplus N^m$.

Proof. It is obvious, by the previous considerations. □

In the following, $sort(\cdot)$ will be the operator that takes any string of non-negative integers and sorts it into weakly increasing order.

Theorem 1.7. *Let* $A = K[X_1, ..., X_n]$ *be a polynomial ring, let* $A^{(s)}$ *be the* $s-$ *Veronese square-free subring of* A*. Then* $A^{(s)} = K[\{T_{i_1 \dots i_s}, 1 \leq i_1 < i_2 < \dots i_s \leq n\}]/I$*, where* I *is generated by the set of binomials*

$$T_{j_1j_2\dots j_s}T_{k_1k_2\dots k_s} - T_{l_1l_2\dots l_s}T_{q_1q_2\dots q_s}$$

$$1 \leq j_1 < j_2 < \dots j_s \leq n$$
$$1 \leq k_1 < k_2 < \dots < k_s \leq n$$
$$1 \leq l_1 < l_2 < \dots < l_s \leq n$$
$$1 \leq q_1 < q_2 < \dots < q_s \leq n$$

with $sort(j_1j_2 \dots j_sk_1k_2 \dots k_s) = sort(l_1l_2 \dots l_sq_1q_2 \dots q_s)$, *Proof.* See [3], Remark 14.1. □

In the lexicographic order for the indexes of the variables $T_{i_1 i_2 \dots i_s}$, put $T_i = T_{i_1 i_2 \dots i_s}$ if i is the place of $s-$tuple $(i_1, i_2, \dots, i_s) \in N^s$ in the lex order. Then $K[\{T_{i_1 i_2 \dots i_s}, 1 \le i_1 < i_2 < \dots i_s \le n\}] = K[T_1, \dots, T_{\binom{n}{s}}]$.
Let $A = K[X_1, ..., X_s]$ and $B = K[Y_1, ..., Y_t]$ be polynomial rings and let $C = K[\{Z_{ij}, 1 \le i \le s, i \le j \le t\}]$ the polynomial ring in st variables.
Given a homogeneous polynomial $f = \sum_{i_1 \le i_2 \le \dots \le i_k} a_{i_1 \dots i_k} X_{i_1} \dots X_{i_k}$ of A and given any sequence of numbers $1 \le j_1 \le j_2 \le \dots \le j_k \le t$, we can define the homogeneous polynomial

$$f_{j_1 \dots j_k} = \sum a_{i_1 \dots i_k} Z_{i_1 j_1} \dots Z_{i_k j_k}$$

in C. In the same way, for any homogeneous polynomial $g = \sum_{j_1 \le j_2 \le \dots \le j_k} b_{j_1 \dots j_k} Y_{i_1} \dots Y_{i_k}$ and for any sequence of numbers $1 \le i_1 \le i_2 \le \dots \le i_k \le s$, we define

$$g_{i_1 \dots i_k} = \sum b_{j_1, \dots, j_k} Z_{i_1, j_1} \dots Z_{i_k, j_k}.$$

Moreover, we consider the lexicographic order.

Theorem 1.8. *Let $A = K[X_1, ..., X_n]$ and $B = K[Y_1, ..., Y_m]$ two polynomial rings and let $A^{(s)}$ be and $B^{(t)}$ be the $s-$Veronese and $t-$Veronese square-free subrings of A and B.*
Then the Segre product $A^{(s)} * B^{(t)}$ is $C = K[\{Z_{ij}, 1 \le i \le \binom{n}{s}, i \le j \le \binom{m}{t}\}]/L$, $K[Z_{ij}]$ the polynomial ring in $\binom{n}{s} \times \binom{m}{t}$ variables, where L is generated by the following set of binomials:

- $\{f^{(i)}_{j_1, j_2}\}, i = 1, ..., p$, $I = (f^{(1)}, ..., f^{(p)})$, $A^{(s)} = K[U_1, ..., U_{\binom{n}{s}}]/I$
- $\{f^{(j)}_{i_1, i_2}\}, j = 1, ..., q$ *where* $J = (g^{(1)}, ..., g^{(q)})$,
  $B^{(t)} = K\left[V_1, ..., V_{\binom{m}{t}}\right]/I$
- $Z_{ij} Z_{kl} - Z_{il} Z_{kj}, 1 \le i \le \binom{n}{s}, i \le j, l \le \binom{m}{t}$.

Proof. See [3]. □

Theorem 1.9. *Let the notations be as in Theorem 1.8 and let $\pi_2 : N^{\binom{n}{s} \times \binom{m}{t}} \to N^n \oplus N^m$ be the map of semigroups rings. Then we have the following facts:*

1. *Consider a relation $g^{(i)}_{i_1 i_2} = Z_{i_1 j_1} Z_{i_2 j_2} - Z_{i_1 j_3} Z_{i_2 j_4}$ that comes from the relation $g^{(i)} = V_{j_1} V_{j_2} - V_{j_3} V_{j_4}$, $j_1 \le j_2, j_3 \le j_4$.*

   The correspondent movie has the elements 1 and -1 on the same i_1th-row (they come from $Z_{i_1 j_1}$ and $Z_{i_1 j_3}$) and the elements 1 and -1 on the same i_2th-row (they come from $Z_{i_2 j_2}$ and $Z_{i_2 j_4}$)(if $i_1 =$

$i_2, 1, -1, 1, -1$ are in the same row).

2. *Consider a relation $f^{(i)}_{j_1,j_2} = Z_{i_1j_1}Z_{i_2j_2} - Z_{i_3j_1}Z_{i_4j_2}$ that comes from the relation $f^{(i)} = U_{i_1}U_{i_2} - U_{i_3}U_{i_4}, i_1 \leq i_2, i_3 \leq i_4$.*

   The correspondent movie has the elements 1 and -1 on the same j_1th-columns (they come from $Z_{i_1j_1}$ and $Z_{i_3j_1}$) and the elements 1 and -1 on the same j_2th-column(they come from $Z_{i_2j_2}$ and $Z_{i_4j_2}$)(if $j_1 = j_2, 1, -1, 1,$ are in the same column).

3. *Consider a relation $Z_{ij}Z_{kl} - Z_{il}Z_{kj}, 1 \leq i, k \leq \binom{n}{s}, 1 \leq j, l \leq \binom{m}{t}$. The correspondent movie contains a minor of order two of the form*

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \qquad \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

Proof. The proof can be obtained working as in Theorem 3.10,[2]. □

Corollary 1.10. *Let $\pi_2 : N^{\binom{n}{s}\times\binom{m}{t}} \to N^m \oplus N^n$. Then the semigroup $\operatorname{Ker} \pi_2 \subset Z^d$ is generated by $\binom{m}{s} \times \binom{n}{t}-$matrices of the following type:*

1. *Only one row is non-zero and it contains two entries equal to 1 and two entries equal to -1, either*
   only 2 rows are non-zero and each one contains 1 and -1.

2. *Only one column is non-zero and it contains two entries equal to 1 and two entries equal to -1, either*
   only 2 columns are not zero and each one contains 1 and -1.

3. *Only a minor of order two is different from zero and it is of the form*

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad \textit{or} \quad \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Proof. It follows from Theorem 1.9. □

2. INITIAL VECTORS OF MOVES

The aim of this number is to calculate the Groebner basis for semigroups of lattice points of N^n.

Definition 2.1. *([3], Chapter 4) Let $\pi : N^n \to Z^d$ be a semigroup homomorphism with respect to the configuration $\mathcal{A} \subset N^n$ and $\hat{\pi} : K[T_1, \ldots, T_n] \to K[t_1, \ldots, t_d]$ the corresponding $K-$algebras homomorphism. Let be $\prec$ a total order on the monomials of $K[\underline{T}]$, $\underline{T} = T_1, \ldots, T_n$ and let $\underline{u} = \underline{u}^+ - \underline{u}^- \in$*

$Z^n, \underline{u} \in \operatorname{Ker}\pi$. *We say that* $in_\prec \underline{u} = \underline{u}^+$ *if* $\underline{T}^{\underline{u}^+} \succ \underline{T}^{\underline{u}^-}$, $in_\prec(\underline{u}) = \underline{u}^-$ *if* $\underline{T}^{\underline{u}^-} \succ \underline{T}^{\underline{u}^+}$.

Proposition 2.2. *Let* $\underline{v}, \underline{w} \in \operatorname{Ker}\pi$, $\underline{v} = \underline{v}^+ - \underline{v}^-$, $\underline{w} = \underline{w}^+ - \underline{w}^-$. *The following facts are equivalent:*

(1) $in_\prec \underline{w} = in_\prec \underline{v} + \underline{u}, \underline{u} \in N^n$
(2) $\underline{T}^{in_\prec \underline{v}} / \underline{T}^{in_\prec \underline{w}}$

For the property 1) *or* 2), *we write*
$in_\prec \underline{v} / in_\prec \underline{w}$.

Proof. Easy. □

Example 2.3. (i) $\underline{v} = (1,-1,0,0) = (1,0,0,0) - (0,1,0,0)$, $\underline{w} = (4,-3,0,0) =$ $(4,0,0,0) - (0,3,0,0)$.
By lex order, $in_\prec \underline{v} = (1,0,0,0)$ *and* $in_\prec \underline{w} = (4,0,0,0)$. *Then* $(1,0,0,0)/(4,0,($ *since* $(4,0,0,0) = (1,0,0,0) + (3,0,0,0)$.

(ii) *For* $\underline{v} = (1,-1,0,0)$ *and* $\underline{w} = (0,4,-3,0)$, $in_\prec \underline{w} = (0,4,0,0)$. *Since* $(0,4,0,0) = (1,0,0,0) + (-1,4,0,0)$, *then it is not true that* $in_\prec \underline{v} / in_\prec \underline{w}$.

Definition 2.4. *Let* $S = < \underline{u}_1, \ldots, \underline{u}_n >$ *a subsemigroup of* Z^n. *We say that* $\{\underline{u}_1, \ldots, \underline{u}_n\}$ *is a Groebner basis for* S, *if* $\forall\, \underline{a} \in S$, $in_\prec \underline{u}_i / in_\prec \underline{a}$ *for some* i, $1 \le i \le n$.

We are interested to Groebner bases of the semigroup $\ker\pi \subset Z^n$, which produce Groebner bases for the toric ideal $I_{\mathcal{A}}$ of the associated toric variety.

Proposition 2.5. *Let* s, t *be positive integers. Let* $\pi : N^{s\times t} \to N^{s+t}$ *the semigroup homomorphism associated to the configuration* $\mathcal{A} = \{e_i \oplus e'_j, 1 \le i \le s, 1 \le j \le t\}$. *Then the set of* $s \times t$*-matrices, where only a* 2×2 *minor is different from zero and it is of the type,*

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \qquad \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix},$$

is a reduced Groebner basis for $\ker\pi$.

Proof. If we consider the toric ideal $I_{\mathcal{A}}$, the set of binomials is a reduced Groebner basis for $I_{\mathcal{A}}$ with respect the sorted order introduced on the monomials of the ring $K[X_{11}, \ldots, X_{st}]$ ([2], Proposition 5.4). The assertion follows by Definition 2.4. □

Proposition 2.6. *Let* π *as in Proposition 2.5. Then* $in_\prec \operatorname{Ker}\pi$ *is generated by* $n \times s$ *matrices such that the first entry different from zero is in the* $i-th$ *row such that* $i < j$, *where* j *is the index of the column where the second entry is different from zero.*

Proof. It follows from the initial monomial of each generator binomial of $I_{\mathcal{A}}$, more precisely the not sorted monomial between the two monomials of the generator binomial. $\square$

Example 2.7. *For $n = m = 3$:*

(1) The Groebner basis for $\ker \pi$ *is the set of matrices*

$$\mathcal{F} = \left\{ \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \right.$$
$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix},$$
$$\left. \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \right\}$$

(2) The semigroup $in_{\prec} ker \pi$ is generated by the matrices:

$$\mathcal{F} = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \right.$$
$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$
$$\left. \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\}.$$

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THE PERIMETER DEVIATION OF A CONVEX SET FROM A POLYGON

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Abstract. Let C_1 and C_2 be two compact convex subsets of the Euclidean plane. We use $\rho^P(C_1, C_2)$ to denote the L_1-distance between C_1 and C_2 in the space of support functions. Let P_n denote a convex polygon with at most n sides. For a given convex set C, there exists a polygon $P_n = P_n(C)$ which minimizes the distance $\rho^P(C, P_n)$. In a previous paper the author established an upper bound to $\rho^P(C, P_n)$ in terms of n and the perimeter p of C, that is attained for any $n \geq 3$ and $p > 0$ if C is a circle. In the present paper we furnish a new proof of this result and state a sufficient condition for uniqueness of the solution. This condition is proved to be satisfied for $n = 3$ and 4 and is very likely satisfied for all $n > 4$.
MSC 2000: 52A40, 52A10.

1 Perimeter deviation and area deviation of convex sets: a review

Approximation of convex bodies frequently occurs in geometric convexity, theory of packing and covering, in geometric algorithms and optimization, as well as in technical problems. Approximation problems have important applications to practical problems of operations research and pattern recognition.

In this paper we shall be concerned with the approximation of a plane convex set by a convex polygon. There are several methods of measuring the distance between two convex sets. We shall confine ourselves to two of them, the perimeter deviation and the area deviation.

Let $\mathcal{C}^2$ be the class of all compact convex and non-empty subsets of the Euclidean plane. The *perimeter deviation* of $C_1, C_2, \in \mathcal{C}^2$ is defined by

$$\varrho^P(C_1, C_2) = 2p\left([C_1, C_2]\right) - p(C_1) - p(C_2), \tag{1}$$

where $[C_1, C_2]$ denotes the convex hull of $C_1 \cup C_2$, and $p(C)$ denotes the perimeter of the set C. If $C_1 \subset C_2$, definition (1) reduces to

$$\varrho^P(C_1, C_2) = p(C_2) - p(C_1), \qquad [C_1 \subset C_2]. \tag{2}$$

The distance ϱ^P may also be written in the form

$$\varrho^P(C_1, C_2) = \int_{S^1} |h(C_1, u) - h(C_2, u)| ds, \tag{3}$$

where S^1 is the boundary of the unit circle with centre O, ds is the element of arclength, u is a unit vector, and $h(C_i, u)$ denotes the support function of C_i in direction u. Formula (3) shows that ϱ^P is the L_1-metric in the space of support functions of elements of $\mathcal{C}^2$. The distance concept expressed by (3) can be extended to higher dimensions. In relation (1), the perimeter has to be replaced by the mean width of the bodies involved (up to a constant factor).

A further distance function is given by

$$\delta_2(C_1, C_2) = \left(\int_{S^1} |h(C_1, u) - h(C_2, u)|^2 ds \right)^{\frac{1}{2}}$$

and defines the L_2-metric in the space of support functions. The distance δ_2 and its extension to higher dimensions do not admit simple geometric interpretations. However, the L_2-metric is especially useful in connection with analytical aspects of the geometry of convex bodies; see H. Groemer's excellent monograph [9].

The *area deviation* (or *symmetric difference metric*) is defined by

$$\varrho^S(C_1, C_2) = a(C_1 \Delta C_2), \tag{4}$$

where the symmetric difference is given by

$$C_1 \Delta C_2 = (C_1 \cup C_2) \backslash (C_1 \cap C_2), \tag{5}$$

and $a(C)$ stands for the area of C. Note that (5) is equivalent to

$$C_1 \Delta C_2 = (C_1 \backslash C_2) \cup (C_2 \backslash C_1).$$

By (4) and (5) we have

$$\begin{aligned} \varrho^S(C_1, C_2) &= a(C_1 \cup C_2) - a(C_1 \cap C_2) \\ &= a(C_1) + a(C_2) - 2a(C_1 \cap C_2). \end{aligned} \tag{6}$$

Now, ϱ^S is a metric on the class of compact, convex sets with non-empty interior. Definition (4) immediately generalizes to all dimensions, with the respective volume in place of the area a.

Let us point out a relation between perimeter deviation and area deviation of two plane convex sets (see [4]).

Let $S^2(R)$ be the surface of a sphere with the radius R. A subset of $S^2(R)$ is said to be *convex* if it contains, with each pair of its points, the smaller arc or a semicircular arc of a great circle joining them. Throughout we shall assume that the convex set K is a closed proper subset of $S^2(R)$ and has non-empty interior. We call such a set a *convex cap* and denote the collection of all convex caps by $\mathcal{K}$. The distance function

$$\delta^S(K_1, K_2) = a(K_1) + a(K_2) - 2a(K_1 \cap K_2) \tag{7}$$

formed after the model of the symmetric difference metric (6), makes $\mathcal{K}$ into a metric space. Note that every convex cap is contained in some closed hemisphere.

Let T be a fixed point of $S^2(R)$, and let H be the open hemisphere with centre T. For any two convex caps $K_1, K_2 \subset H$ we define a perimeter deviation

$$\delta^P(K_1, K_2) = 2p\left([K_1, K_2]\right) - p(K_1) - p(K_2) \tag{8}$$

(cf. (1)). Let K^* be the polar set of K. Then K_1^*, K_2^* and $K_1^* \cap K_2^*$ are convex caps and

$$[K_1, K_2]^* = K_1^* \cap K_2^*. \tag{9}$$

From (7), (8), (9) and the well-known formula

$$a(K^*) + R\,p(K) = 2\pi R^2 \tag{10}$$

it follows that

$$R\,\delta^P(K_1, K_2) = \delta^S(K_1^*, K_2^*). \tag{11}$$

In the plane touching $S^2(R)$ at the point T let C_1 and C_2 be two compact convex sets with interior points. The radial projection onto $S^2(R)$ maps C_1, C_2 onto two convex caps $K_1, K_2 \subset H$. Simple calculations and an additional approximation argument show that

$$\varrho^S(C_1, C_2) = \lim_{R\to\infty} \delta^S(K_1, K_2) \tag{12}$$

and

$$\varrho^P(C_1, C_2) = \lim_{R\to\infty} \delta^P(K_1, K_2). \tag{13}$$

The combination of (13) with (11) yields

$$\varrho^P(C_1, C_2) = \lim_{R\to\infty} \frac{1}{R}\,\delta^S(K_1^*, K_2^*). \tag{14}$$

For a review of various concepts of deviation between convex bodies in E^d, including a comprehensive bibliography, we refer to the paper [10] and the handbook article [11], both by P.M. Gruber.

Let us now turn to problems of approximation of convex sets by polygons.

We consider the class of all closed convex subsets of the plane with areas and perimeters bounded by given constants. Which set of this class has *minimal area deviation* or *minimal perimeter deviation* from an n-gon?

Let $n \geq 3$ be an integer and let a and p be positive constants such that

$$4\pi \leq p^2/a < 4n\tan(\pi/n),$$

where the last number is the isoperimetric ratio of the regular n-gon. Let $\mathcal{C}(a,p)$ be the class of all compact convex sets with area not less than a and perimeter not greater than p. Furthermore, let $\mathcal{P}_n$ be the class of all convex m-gons with $m \leq n$. The problem to find the *minimum of* $\varrho^S(C,P)$ and the *minimum of* $\varrho^P(C,P)$ for $C \in \mathcal{C}(a,p)$, $P \in \mathcal{P}_n$ was considered in a series of papers by L. Fejes Tóth, A. Heppes, G. Fejes Tóth and A. Florian. In these articles the following three assumptions have been distinguished: (i) $P \supset C$, or (ii) $P \subset C$, or (iii) the mutual position of C and P is not restricted in any way. It has been proved that in each case the pair (C,P) is uniquely determined up to isometry, furthermore $a(C) = a$, $p(C) = p$, and only one of the following three configurations satisfies the minimum requirement:

(a) The set C is a *smooth regular n-gon* which is defined to be a smooth convex set obtained from a regular n-gon P by rounding off its corners with congruent circular arcs. The n-gon P is called the *case* of C.

(b) The set C is a *regular arc-sided n-gon* which is defined as a convex set obtained from a regular n-gon P by joining each two adjacent vertices of P by congruent circular arcs. The n-gon P is called the *kernel* of C.

(c) The set C is an *outer parallel domain of a regular arc-sided n-gon* and P is a certain regular n-gon assigned to C.

Note that the configurations described in (a) and (b) may be regarded as limit cases of that in (c). For a detailed discussion of the results and references see the handbook article [7]. The above theorems of approximation have important applications to packing and covering of the plane with sets of equal area or equal perimeter.

The notions of smooth regular n-gon and regular arc-sided n-gon can be transferred to the sphere. Some approximation theorems analogous to those in the plane and under the restriction $P \supset C$ or $P \subset C$ can be established [5]. Since these theorems are equivalent by spherical polarity, the situation on S^2 appears in a way more satisfactory than that in the plane. It would also be desirable to extend such theorems to S^2 where the mutual position of a given convex set C and an approximating polygon P is not restricted.

Let us now direct our attention to another group of approximation theorems.

Let C be a compact convex set of area a, and for $n \geq 3$ let $a_n(C)$ denote the maximum of the areas of all convex polygons in C with at most n vertices. Then

$$a_n(C) \geq \frac{n}{2\pi} \sin \frac{2\pi}{n} \, a, \tag{15}$$

and equality holds only if C is an ellipse [2], [9]. E. Sas proved inequality (15) in an elegant analytic way. This theorem confirms a conjecture by L. Fejes Tóth who supplied the proof of uniqueness, making use of Fourier series.

Furthermore, let $a_n(C)$ be as above and let $A_n(C)$ be the minimum of the areas of all convex n-gons containing C. Then

$$\frac{a_n(C)}{A_n(C)} \geq \cos^2 \frac{\pi}{n} \tag{16}$$

with equality if C is an ellipse. The nice inequality (16) is due to D. Lazar [12], [2]. His proof is based on a theorem concerning a pair of convex n-gons. L. Fejes Tóth gave to this theorem the following modified version: "On each side of a convex n-gon A choose a point so that the triangles cut off by the sides of the n-gon B determined by these points have equal area. Then $a(B)/a(A)$ attains its minimum if A is affinely regular and the vertices of

B are mid-points of the sides of A". Fejes Tóth applied the theorem to the polygonal approximation of convex compact sets with given area and affine perimeter not less than a given positive constant [3].

Corresponding problems for the perimeter instead of the area have been solved by R. Schneider [14]. Let C be a compact convex set of perimeter L. For $n \geq 3$ let $l_n(C)$ denote the maximum of the perimeters of all convex n-gons contained in C, and let $L_n(C)$ be the minimum of the perimeters of all convex n-gons containing C. Then

$$L_n(C) \leq \frac{n}{\pi} \tan \frac{\pi}{n} L, \tag{17}$$

$$l_n(C) \geq \frac{n}{\pi} \sin \frac{\pi}{n} L. \tag{18}$$

In both cases equality holds only if C is a circle. The uniqueness statement for (17) has been solved by Schneider. The uniqueness problem for (18) leads to the task of showing that the equation

$$\tan(k\frac{\pi}{n}) = k \tan \frac{\pi}{n} \qquad [n \geq 3] \tag{19}$$

has no solution in integers $k \geq 2$. This has been proved by Florian and Prachar [8].

Let a compact convex set C and an integer $n \geq 3$ be given, and let P_n be a convex polygon with at most n sides. In the following the mutual position of C and P_n will *not be restricted in any way.* There exists some P_n which will be denoted by $P_n(C)$, such that

$$\varrho^P(C, P_n(C)) = \inf \varrho^P(C, P_n) \equiv \varrho_n(C), \tag{20}$$

where the infimum extends all over $P_n \in \mathcal{P}_n$. If C has the perimeter p, the author has shown that

$$\varrho_n(C) \leq p \left(1 - \frac{2n}{\pi} \arcsin(\frac{1}{2} \sin \frac{\pi}{n})\right) \tag{21}$$

with equality if C is a circle [6]. If K is a circle of perimeter p, $P_n(K)$ is a regular n-gon concentric with K. The vertices of $P_n(K)$ are outside K, and the edges of $P_n(K)$ intersect the interior of K. Let A be a vertex of $P_n(K)$, and let $2\tau_n$ denote the outer angle of $[K, P_n(K)]$ at A. Then

$$\sin \tau_n = \frac{1}{2} \sin \frac{\pi}{n} \tag{22}$$

(see Lemma 8 and Corollary 5 of [6]).

Ingredients of the mentioned proof of (21) are repeated Blaschke symmetrization of C, the fact that Blaschke symmetrization does not diminish $\varrho_n(C)$, and a selection theorem. This approach does *not* permit to show that the circle is the only set satisfying (21) with equality.

In Section 2 we shall outline the proof of a theorem that furnishes more than inequality (21).

Theorem. *Let C be a compact convex set of perimeter p with interior points, and let $n \geq 3$ be an integer. There exists a one-parametric family of convex n-gons such that the mean perimeter deviation from C of these polygons satisfies inequality (21). Let*

$$\tau_n = \arcsin(\frac{1}{2} \sin \frac{\pi}{n}) < \frac{\pi}{2}. \tag{23}$$

If the equation

$$\frac{\sin k(\frac{\pi}{n} + \tau_n)}{\sin k(\frac{\pi}{n} - \tau_n)} = \frac{\sin(\frac{\pi}{n} + \tau_n)}{\sin(\frac{\pi}{n} - \tau_n)} \tag{24}$$

has no solution in integers $k \geq 2$, then equality holds in (21) if and only if C is a circle.

2 Proof of the Theorem

Let C be a compact convex set of perimeter p with interior points, and let $n \geq 3$ be an integer. Let σ and τ be positive numbers such that

$$\sigma + \tau = \frac{\pi}{n}. \tag{25}$$

For some $\alpha \in [0, 2\pi]$ we consider the $2n$ support lines $s_1, s_2, s_3, \ldots, s_{2n}$ of C, whose outer normals form the following angles with a fixed direction:

$$\alpha,\ \alpha + 2\sigma,\ \alpha + 2\sigma + 2\tau,\ \alpha + 2 \cdot 2\sigma + 2\tau,\ \alpha + 2 \cdot 2\sigma + 2 \cdot 2\tau,$$
$$\ldots, \alpha + (n-1) \cdot 2\sigma + (n-1) \cdot 2\tau,\ \alpha + n \cdot 2\sigma + (n-1) \cdot 2\tau.$$

Let $s_{i-1} \cap s_i = A_i$, for $i = 1, \dots, 2n$ where $s_0 = s_{2n}$. The $A_i (i = 1, \dots, 2n)$ are the vertices of a closed convex $2n$-gon circumscribed about C, which we shall denote by $Q_{2n}(\alpha)$. Let $P_n(\alpha)$ be the convex n-gon whose vertices are the *odd-numbered* points $A_1, A_3, \dots, A_{2n-1}$. Clearly we have

$$\varrho_n(C) \le \inf_{0 \le \alpha \le 2\pi} \varrho^P(C, P_n(\alpha)). \tag{26}$$

The perimeter deviation $\varrho^P(C, P_n(\alpha))$ depends on the perimeters of $[C, P_n(\alpha)]$ and $P_n(\alpha)$. If K is a compact convex set with the support function $h(K)$, then

$$p(K) = \int_0^{2\pi} h(K, \varphi) d\varphi. \tag{27}$$

If K is the convex hull of some sets $K_1, \dots, K_m$, then

$$h(K, \varphi) = \max\{h(K_1, \varphi), \dots, h(K_m, \varphi)\} \tag{28}$$

for $0 \le \varphi \le 2\pi$. Observe that $[C, P_n(\alpha)]$ is the convex hull of the sets C, $\{A_1\}$, $\{A_3\}, \dots, \{A_{2n-1}\}$. Applying (27) and (28) to $[C, P_n(\alpha)]$ we obtain by some calculation

$$\begin{aligned} p([C, P_n(\alpha)] &= \sum_{k=0}^{n-1} \int_{\alpha + k\frac{2\pi}{n}}^{\alpha + k\frac{2\pi}{n} + 2\sigma} h(\varphi) d\varphi \\ &+ \tan\tau \sum_{k=0}^{n-1} [h(\alpha + k\frac{2\pi}{n}) + h(\alpha + k\frac{2\pi}{n} + 2\sigma)], \end{aligned} \tag{29}$$

where $h(\varphi) = h(C, \varphi)$.

It can easily be shown that

$$\overline{A_1A_3} \ge (\overline{A_1A_2} + \overline{A_2A_3}) \cos\sigma \tag{30}$$

with equality if and only if $\overline{A_1A_2} = \overline{A_2A_3}$, and similarly for $\overline{A_3A_5}, \dots, \overline{A_{2n-1}A_1}$. Adding the inequalities we find

$$p(P_n(\alpha)) \ge p(Q_{2n}(\alpha)) \cos\sigma. \tag{31}$$

Note that $Q_{2n}(\alpha)$ is the convex hull of the sets $\{A_1\}, \{A_2\}, \dots, \{A_{2n}\}$. Making use of (27) and (28) once more, we obtain

$$p(Q_{2n}(\alpha)) = \frac{\sin\frac{\pi}{n}}{\cos\sigma\cos\tau} \sum_{k=0}^{n-1} [h(\alpha + k\frac{2\pi}{n}) + h(\alpha + k\frac{2\pi}{n} + 2\sigma)]. \tag{32}$$

From (1), (29), (31) and (32) it follows that

$$\begin{aligned} \varrho^P(C, P_n(\alpha)) &\leq 2\sum_{k=0}^{n-1} \int_{\alpha+k\frac{2\pi}{n}}^{\alpha+k\frac{2\pi}{n}+2\sigma} h(\varphi)d\varphi - p \\ &+ (2\tan\tau - \frac{\sin\frac{\pi}{n}}{\cos\tau})\sum_{k=0}^{n-1}[h(\alpha + k\frac{2\pi}{n}) + h(\alpha + k\frac{2\pi}{n} + 2\sigma)]. \end{aligned} \tag{33}$$

By (30), equality holds if and only if

$$\overline{A_{2i-1}A_{2i}} = \overline{A_{2i}A_{2i+1}} \tag{34}$$

for $i = 1, \ldots, n$.

To obtain an upper bound on $\varrho(C)$ that is independent of α we remark that

$$\inf_{0\leq\alpha\leq 2\pi} \varrho^P(C, P_n(\alpha)) \leq \frac{1}{2\pi}\int_0^{2\pi} \varrho^P(C, P_n(\alpha))d\alpha \tag{35}$$

with equality if and only if $P_n(\alpha)$ is a constant. Changing the order of the integrations one deduces from (33) that

$$\frac{1}{2\pi}\int_0^{2\pi} \varrho^P(C, P_n(\alpha))d\alpha \leq p\left(1 - \frac{2n}{\pi}\tau + \frac{n}{\pi}(2\tan\tau - \frac{\sin\frac{\pi}{n}}{\cos\tau})\right). \tag{36}$$

Equality holds if and only if

$$\overline{A_1A_2} = \overline{A_2A_3} \tag{37}$$

for *all* $\alpha \in [0, 2\pi]$. The function of τ on the right-hand side of (36) attains its minimum exactly for

$$\tau = \arcsin\left(\frac{1}{2}\sin\frac{\pi}{n}\right)$$

which coincides with τ_n defined by (23). If we choose

$$\tau = \tau_n$$

as we shall do in the following, then (36) takes the form

$$\frac{1}{2\pi}\int_0^{2\pi} \varrho^P(C, P_n(\alpha))d\alpha \leq p\left(1 - \frac{2n}{\pi}\arcsin(\frac{1}{2}\sin\frac{\pi}{n})\right), \tag{38}$$

as required by the Theorem. Equality holds if and only if condition (37) is satisfied for all $\alpha \in [0, 2\pi]$. Note that inequality (21) is a consequence of (26), (35) and (38). Equality holds in each case if C is a circle.

We let follow some remarks on τ_n. From the definition (23) it follows that

$$0 < \tau_n < \frac{\pi}{2n}. \tag{39}$$

Since $\sin \tau_n = \frac{1}{2} \sin \frac{\pi}{n}$ is algebraic, one sees that τ_n is transcendental. Observe that (23) can be written in the form

$$2\cos(\frac{\pi}{2} - \tau_n) - \cos(\frac{\pi}{2} - \frac{\pi}{n}) = 0. \tag{40}$$

Theorem 7 of a paper by Conway and Jones [1] implies that (40) is *not* satisfied if τ_n is a rational multiple of π. Hence, we may state for later use that

$$\frac{\tau_n}{\pi} \quad \text{is irrational.} \tag{41}$$

By means of the Gelfond-Schneider theorem (which solved the seventh problem of Hilbert's list) it can be deduced from (41) that $\frac{\tau_n}{\pi}$ is even transcendental.

Let us assume that a convex set C of perimeter p satisfies (21) with equality. Hence equality also must hold in (38). We have seen that this is the case if and only if (37) is satisfied, i.e.,

$$\overline{A_1A_2} = \overline{A_2A_3}$$

for all $\alpha \in [0, 2\pi]$. Condition (37) can be stated in terms of the support function $h(C, \varphi) = h(\varphi)$ in the following way.

Let the origin be an interior point of C, and let t_1, t_2, t_3 be three support lines of C with outer normals in the direction ω, $\omega+\alpha$, $\omega+\alpha+\beta$, respectively. Here α and β are positive and $\alpha + \beta < \pi$. Let $T' = t_1 \cap t_2$, $T'' = t_2 \cap t_3$ and $s = \overline{T'T''}$. Then

$$\begin{aligned} s \sin\alpha \sin\beta &= h(\omega)\sin\beta + h(\omega+\alpha+\beta)\sin\alpha \\ &- h(\omega+\alpha)\sin(\alpha+\beta); \end{aligned} \tag{42}$$

see [13], p. 312. If (42) is applied to $s = \overline{A_1A_2}$ and $s = \overline{A_2A_3}$, then (37) implies that

$$[h(\alpha) - h(\alpha + 2\sigma_n)](\sin 2\tau_n + \sin \frac{2\pi}{n}) + [h(\alpha + \frac{2\pi}{n}) - h(\alpha - 2\tau_n)] \sin 2\sigma_n = 0, \tag{43}$$

where $\sigma_n = \frac{\pi}{n} - \tau_n$ (see (25)). We are looking for such solutions h of the functional equation (43) which are continuous and periodic with 2π.

Let

$$h(\alpha) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos k\alpha + b_k \sin k\alpha) \tag{44}$$

be the Fourier series of h. Making use of (44) one obtains the Fourier expansion of the function on the left-hand side of (43). Since this function vanishes identically, all coefficients of the new Fourier series are equal to zero. This yields a system of two linear homogeneous equations for each pair $a_k, b_k (k = 1, 2, \ldots)$. It is important to remark that, by (41), τ_n and σ_n are *irrational* multiples of π. Based on this fact it can be shown that the determinant of the system vanishes if and only if

$$\frac{\sin k(\frac{\pi}{n} + \tau_n)}{\sin k(\frac{\pi}{n} - \tau_n)} = \frac{\sin(\frac{\pi}{n} + \tau_n)}{\sin(\frac{\pi}{n} - \tau_n)}. \tag{45}$$

This is clearly true for $k = 1$. If it could be proved that (45) has *no solution* in integers $k \geq 2$, then we would have $a_k = b_k = 0$ for $k = 2, 3, \ldots$. Thus, the function $h(\alpha)$ would reduce to $h(\alpha) = \frac{a_0}{2} + a_1 \cos \alpha + b_1 \sin \alpha$, i.e., the support function of a circle.

This proves the second statement of the Theorem. □

In the following, we shall prove for $n = 3$ and $n = 4$ that equation (45) does not have a solution in integers $k \geq 2$. Furthermore, we shall advance some arguments to support *the conjecture* that this is also true for any $n \geq 5$.

If $k \geq 2$ satisfies (45), in view of (41) we have $\cos(k\frac{\pi}{n}) \neq 0$ and $\sin(k\frac{\pi}{n}) \neq 0$. It is easy to show that (45) is equivalent to

$$\frac{\tan k\tau_n}{\tan \tau_n} = \frac{\tan k\frac{\pi}{n}}{\tan \frac{\pi}{n}}. \tag{46}$$

Let us assume that the integer k' satisfies (46). Then $k \equiv k' \bmod n$, $k \neq k'$ is not a solution of (46), and the same holds for $k \equiv -k' \bmod n$, $k \neq -k'$. Hence, $k \equiv \frac{n}{2} \bmod n$ (if n is even) is not a solution.

Let $N(n)$ be the number of integers $k \geq 2$ satisfying (46). The above simple remarks imply that

$$N(n) \leq \lfloor \frac{n-3}{2} \rfloor, \qquad (n \geq 3) \tag{47}$$

whence

$$N(3) = N(4) = 0, \tag{48}$$

as desired. Since $N(n)$ is finite for any $n \geq 3$, the Fourier series of $h(\varphi)$ is a trigonometric polynomial and possesses derivatives of every order. Particularly, the boundary of an extremal set C does not contain straight segments, i.e. C is a strictly convex domain. Observe that this conclusion results by the simple argument that τ_n is an irrational multiple of π. It may be that a possible proof of the conjecture $N(n) = 0$ for $n \geq 5$ requires employing particular properties of τ_n.

J. Linhart kindly made a computer search of integer solutions of (45) which produced the following result:
If $3 \leq n \leq 2^{10} = 1024$, then there is no solution of (45) with $2 \leq k \leq 1024$.□

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q-paths of a graph and incidence matrix II

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Abstract

This work studies, in the degree $q \leqslant 6$, the paths of length $(q-1)$ of a connected graph G and the generators of the corresponding generalized graph ideal $I_q(G)$, using only the incidence matrix of G.

Such a problem is known for generalized graph ideals when $q \leqslant 4$.

The extension to higher degrees is made by applying different techniques of computation in the proofs.

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1 Introduction

Algebraically speaking, determining some paths of length $(q-1)$, q positive integer, of a connected graph G, means to find generators of a monomial ideal to which G can be associated, the generalized graph ideal $I_q(G)$.

The problem of computing, using only the incidence matrix of G, the number and structure of paths of fixed length in G, and the generators of the relative generalized graph ideal, has been presented and justified in [2].

There, an answer to it has been given in the degree $q \leqslant 4$. The number of paths of length 2 or 3 in G has been obtained in terms of multiplicity of pairs of rows in the incidence matrix of G. Their structure has been found by joining alternatively on the rows and columns of the incidence matrix specific entries 1 relative to the vertices of the paths beginning from the inside, that is either from a combination of pairs of 1's on the rows or from the two 1's on the columns of the matrix, respectively.

In this paper, by also introducing new strategies in the calculations, the problem is solved up to $q = 6$. The extension method is not at all simple, especially to get the number of paths of length 4 and 5 for the graph G, because of possible triangular and squared cycle subgraphs contained in it.

2 Definitions and known results

Definition 1 A graph G is said *connected* if every pair of vertices of G are joined by a path, that is a walk whose vertices are distinct.

Definition 2 Let G be a connected graph having vertices $v_1, \ldots, v_p$. A *generalized graph ideal* $I_q(G)$, $\mathbb{N} \ni q \leqslant p$, is an ideal of the polynomial ring $K[x_1, \ldots, x_p]$, where K is a field and each variable x_i corresponds to v_i, generated by all the square-free monomials $x_{i_1} \cdots x_{i_q}$ of degree q such that the vertex v_{i_j} is adjacent to $v_{i_{j+1}}$, for all $1 \leqslant j \leqslant (q-1)$.

Remark 1 $I_2(G)$ is the generalized graph ideal generated by the edges of G, the so-called edge ideal. More generally, the generators of $I_q(G)$ are paths of G of length $(q-1)$, simply called $(q-1)$-paths.

Definition 3 A monomial ideal $L_q \subset K[x_1,\ldots,x_m; y_1,\ldots,y_n]$ is an *ideal of mixed products* if it is writable as $L_q = I_p J_r + I_s J_t$, where

$q = p + r = s + t$, for p, r, s, t non-negative integers;

I_p (resp. J_r) is an ideal of $K[x_1,\ldots,x_m; y_1,\ldots,y_n]$ generated by square-free monomials of degree p (resp. r) in the variables $x_1,\ldots,x_m$ (resp. $y_1,\ldots,y_n$).

Definition 4 A graph G is said *bipartite* if the vertex set V of G can be partitioned into two subsets V_1 and V_2 such that every edge of G joins V_1 with V_2.

For bipartite graphs, the generalized graph ideals are particular ideals of mixed products. To this proposal the following properties hold.

Property 1 *Let G be a complete bipartite graph having vertex set $V = \{x_1,\ldots,x_m; y_1,\ldots,y_n\}$. Then, for $2 \leqslant q < (m+n)$, the generalized graph ideal $L_q(G)$ is of the form*

$$L_q(G) = \begin{cases} I_h J_{h+1} + I_{h+1} J_h & \text{if } q = 2h+1 \\ I_h J_h & \text{if } q = 2h \end{cases} .$$

Property 2 *Let $L_q \subset K[x_1,\ldots,x_m; y_1,\ldots,y_n]$, $2 \leqslant q < (m+n)$, be an ideal of mixed products of the form* *a)* $I_h\, J_{h+1}$, *or* $I_{h+1}\, J_h$, *or* $I_h\, J_{h+1} + I_{h+1}\, J_h$ *for* $h = \dfrac{q-1}{2}$;

b) $I_h\, J_h$ *for* $h = \dfrac{q}{2}$.

Then $L_q = L_q(G)$, *where* G *is a complete bipartite graph with vertex set* V.

Definition 5 Let G be a graph with p vertices and t edges. The *incidence matrix* M_G of G is a $(p \times t)$-matrix whose entries a_{ij} are equal to 1 if the i-th vertex of G belongs to the j-th edge, 0 otherwise.

Remark 2 Each row of M_G has as many entries 1 as the degree of the relative vertex of G. Each column of M_G has two entries 1 and the remaining are 0.

Definition 6 Let G be a connected graph and M_G be the incidence matrix of G. We call *multiplicity* of a pair of rows in M_G, corresponding in G to a pair of vertices having the degrees α and β respectively, the product $(\alpha-1)\,(\beta-1)$, and denote it $\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$.

Property 3 *Let* G *and* M_G *be as in the last definition. The multiplicity of a pair of rows in* M_G, *that correspond to a pair of vertices of* G *joined by a* $(q-1)$*-path,* $q \geqslant 2$, *and that have the degrees* $\alpha \geqslant 2$ *and* $\beta \geqslant 2$, *gives the number of walks of length* $(q+1)$ *in* G *containing the* $(q-1)$*-path inside.*

To compute $(q-1)$-paths of G and the generators of $I_q(G)$, when $q = 3, 4$, the following results hold.

Proposition 1 *A connected graph* G *with* $p \geqslant 3$ *vertices has* $\displaystyle\sum_{i=1}^{p} \binom{\lambda_i}{2}$ 2*-paths, where* λ_i *denotes the number of entries* 1 *in the* i*-th row of the incidence matrix* M_G .
On each row of M_G *let's consider every pair* $a_{i_2\, h}$, $a_{i_2\, k}$ *of entries* 1 , *and let* $a_{i_1\, h}$, $a_{i_3\, k}$ *be the other entries* 1 *of the relative columns.*
Then all the 2*-paths of* G *and the generators of* $I_3(G)$ *are of the type* $x_{i_1}\, x_{i_2}\, x_{i_3}$, $1 \leqslant i_1 \neq i_2 \neq i_3 \leqslant p$.

Proof. See [2] , Theorem 2 .

Proposition 2 *A connected graph* G *with* $p \geqslant 4$ *vertices,* t *edges, and* s *cycle subgraphs* C_3, *has* $\displaystyle\sum_{j=1}^{t} \begin{bmatrix}\alpha_j\\ \beta_j\end{bmatrix} - 3\,s$ 3*-paths, where* $\begin{bmatrix}\alpha_j\\ \beta_j\end{bmatrix}$, $\alpha_j \geqslant \beta_j \geqslant 2$,

denote the multiplicity of the rows of the incidence matrix M_G on which the entries 1 of its j-th column lie.
If $a_{i_2\,j}$, $a_{i_3\,j}$ are such entries, let's combine the pairs of 1's on the row R_{i_2} that contain $a_{i_2\,j}$ together with the pairs of 1's on R_{i_3} that contain $a_{i_3\,j}$, and let $a_{i_2\,h}$, $a_{i_3\,k}$ be the other entries 1 in the pairs for each of these combinations, $a_{i_1\,h}$, $a_{i_4\,k}$ be the remaining entries 1 of the relative columns.
Then all the 3-paths of G are of the type $x_{i_1}\,x_{i_2}\,x_{i_3}\,x_{i_4}$, $1 \leqslant i_1 \neq i_2 \neq i_3 \neq \neq i_4 \leqslant p$, and the generators of $I_4(G)$ are the 3-paths of G different from one another for at least an index.

Proof. See [2] , Theorem 3 .

3 New cases

In this section the problem of computing up to $q=6$ the number and the structure of the $(q-1)$-paths of a connected graph G and the generators of the corresponding generalized graph ideal $I_q(G)$ is developed.

– 4-paths of G and generators of $I_5(G)$

Proposition 3 *Let G be a connected graph with $p \geqslant 5$ vertices $v_1, \dots, v_p$, t edges, s cycle subgraphs C_3 , r cycle subgraphs C_4 , and incidence matrix M_G .*
Let d_h be the number of vertices of G adjacent to C_3, $h = 1, \dots, s$.
For every $(p \times 2)$-submatrix A_ℓ of M_G with only one row of 1's, let $\begin{bmatrix} \alpha_\ell \\ \beta_\ell \end{bmatrix}$, $\alpha_\ell \geqslant \beta_\ell \geqslant 2$, be the multiplicity of the rows of M_G corresponding to the rows of A_ℓ with a unique entry 1 .
Then G has $\displaystyle\sum_\ell \begin{bmatrix} \alpha_\ell \\ \beta_\ell \end{bmatrix} - \sum_{h=1}^{s} (\,3 + 2\,d_h) - 4\,r$ *4-paths.*

Proof. A 4-path of G can be thought of as a pair of 3-paths having a common 2-path and the remaining two edges without a common vertex.
According to Proposition 1 , the internal vertex and the ends of any 2-path of G characterize a $(p \times 2)$-submatrix A_ℓ of M_G having only one row of 1's, and these vertices correspond in A_ℓ to the row of 1's and to the pair of rows with a unique entry 1, respectively.
The number of such submatrices is $\displaystyle\sum_{i=1}^{p} \binom{\deg v_i}{2}$.
If m_ℓ is the multiplicity of the rows in M_G that correspond to the rows of A_ℓ with a unique entry 1 , every pair of 3-paths having a common 2-path

determine m_ℓ walks of length 4 in G having as internal vertices the three vertices of their common 2-path.

For $\ell = 1, \dots, \sum_{i=1}^{p} \binom{\deg v_i}{2}$, all these walks of length 4 in G are found. The assertion follows excluding walks having some repeated vertex, that is

– for every cycle subgraph C_3 of G, there are

  3 distinct walks, having the same external edges, and

  twice the sum of the degrees of the vertices of C_3 minus twice the sum of the degrees of the vertices of a triangular cycle graph distinct walks, having one pair of equal vertices not at both the ends,

– for every cycle subgraph C_4 of G, there are

  4 distinct walks, having the same ends.

Theorem 1 *Let G, M_G, and A_ℓ be as in the Proposition 3, and R_{i_3} be the row of 1's in any A_ℓ.*
Let R_{i_2}, R_{i_4} denote the rows of M_G relative to the rows of A_ℓ with a unique entry 1, and R_{i_1}, R_{i_5} be the rows of M_G on which the remaining entry 1 of the columns, not belonging to A_ℓ, located by an entry 1 in R_{i_2} and an entry 1 in R_{i_4} lies.
Then all the 4-paths of G are of the type $x_{i_1} x_{i_2} x_{i_3} x_{i_4} x_{i_5}$, $1 \leqslant i_1 \neq i_2 \neq \neq i_3 \neq i_4 \neq i_5 \leqslant p$, where the vertices correspond to the above rows of M_G, and the generators of $I_5(G)$ are the 4-paths of G different from one another for at least an index.

Proof. To construct 4-paths in G, let's start from a 2-path whose middle vertex is given by the row R_{i_3} of 1's in any $(p \times 2)$-submatrix A_ℓ of M_G, and whose ends by the two rows of A_ℓ with a unique 1.
These three vertices represent the internal vertices of the 4-paths can be obtained from A_ℓ.
To determine the ends of such 4-paths, let's consider all the entries 1 lying on each of the rows R_{i_2}, R_{i_4} of M_G relative to the rows of A_ℓ with a unique entry 1. If one of these rows in M_G contains only the entry 1 of the correspondent row in A_ℓ, no 4-path is formed.
Otherwise, let S_n, $n \geqslant 1$, denote every set whose elements are two pairs of entries 1, a pair on R_{i_2}, the other one on R_{i_4}, such that an entry of each pair always lies on A_ℓ. If Γ_h and Γ_k, $h \neq k$, are the columns of M_G to which the entry not lying on A_ℓ in each pair of any S_n belongs, let R_{i_1}, R_{i_5} be the rows of M_G on which the remaining entry 1 of Γ_h and Γ_k lies.

When R_{i_1}, R_{i_5} are different from each other and from R_{i_2}, R_{i_3}, R_{i_4}, they give the ends of the 4-paths in G that come from A_ℓ.
Such 4-paths, for every choice of A_ℓ, have the form $x_{i_1}\, x_{i_2}\, x_{i_3}\, x_{i_4}\, x_{i_5}$.
The last assertion derives from the definition of generalized graph ideal.

– 5-paths of G and generators of $I_6(G)$

Proposition 4 *Let G be a connected graph with $p \geqslant 6$ vertices $v_1, \ldots, v_p$, t edges, s cycle subgraphs C_3, r cycle subgraphs C_4, ρ cycle subgraphs C_5, and incidence matrix M_G.*
For any C_3 in G, let d_h be the number of vertices v_{i_λ} of G adjacent to C_3, $h = 1, \ldots, s$, and $\wp$ be the number of pairs of C_3 with a common edge.
For any C_4 in G, let δ_k be the number of vertices of G, not belonging to C_4, adjacent to C_4, $k = 1, \ldots, r$.
For every $(p{\times}3)$-submatrix B_ℓ of M_G having exactly two rows both with two 1's, let $\begin{bmatrix}\alpha_\ell\\ \beta_\ell\end{bmatrix}$, $\alpha_\ell \geqslant \beta_\ell \geqslant 2$, be the multiplicity of the rows of M_G corresponding to the rows of B_ℓ with a unique entry 1. Then the number of 5-paths of G is

$$\sum_\ell \begin{bmatrix}\alpha_\ell\\ \beta_\ell\end{bmatrix} - 2\left(\sum_{\lambda=1}^{d_h} (\deg v_{i_\lambda} - 1)\right) + 2\,\wp - \sum_{k=1}^{r} (4 + 2\,\delta_k) - 5\,\rho\,.$$

Proof. A 5-path of G can be thought of as a pair of 4-paths having a common 3-path and the remaining two edges without a common vertex.
According to Proposition 2, the two internal vertices and the ends of any 3-path of G characterize a $(p{\times}3)$-submatrix B_ℓ of M_G having exactly two rows both with two 1's, and these vertices correspond in B_ℓ to the rows with two 1's and to the pair of rows with a unique entry 1, respectively.
The number of such submatrices is $\sum n_j - 3\,s$, where n_j is the multiplicity of the rows of M_G on which the entries 1 of its j-th column lie, $j = 1, \ldots, t$.
If m_ℓ is the multiplicity of the rows in M_G that correspond to the rows of B_ℓ with a unique entry 1, every pair of 4-paths having a common 3-path determine m_ℓ walks of length 5 in G having as internal vertices the four vertices of their common 3-path.
For $\ell = 1, \ldots, \sum n_j - 3\,s$, all these walks of length 5 in G are found. The assertion follows excluding walks with some repeated vertex. In particular

- for every cycle subgraph C_3 of G, by considering all the vertices of G adjacent to each node of C_3, there are

  twice the sum of the degrees of such vertices minus 1 distinct walks, having at least one pair of equal vertices not at both the ends,

but if a vertex of G is adjacent to a pair of nodes of C_3, another triangular cycle subgraph of G that has a common edge with C_3 is formed, so 2 walks, whose middle edge is the one in common, are obtained twice, in the procedure of C_3 as well as in the other one,
then for each pair of cycle subgraphs C_3 of G with a common edge, 2 walks from the above computation are needed to be taken off;

– for every cycle subgraph C_4 of G, there are

  4 distinct walks, having the same external edges, and

  twice the sum of the degrees of the vertices of C_4 minus twice the sum of the degrees of the vertices of a squared cycle graph distinct walks, having at least one pair of equal vertices not at both the ends,

but if an edge of G joins two non-consecutive vertices of C_4, a pair of cycle subgraphs C_3 contained in C_4 arise, so 4 walks, having two pairs of equal vertices, are the same walks obtained in the single procedures of such C_3 ,
then for each of the above edges of G, 4 walks from the above computation are needed to be taken off;

– for every cycle subgraph C_5 of G, there are

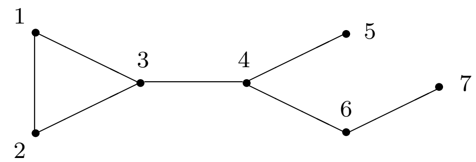
  5 distinct walks, having the same ends.

Theorem 2 *Let G, M_G, and B_ℓ be as in the Proposition 4, and R_{i_3}, R_{i_4} be the rows both with two 1's in any B_ℓ.*
Let R_{i_2}, R_{i_5} denote the rows of M_G relative to the rows of B_ℓ with a unique entry 1, and R_{i_1}, R_{i_6} be the rows of M_G on which the remaining entry 1 of the columns, not belonging to B_ℓ, located by an entry 1 in R_{i_2} and an entry 1 in R_{i_5} lies.
Then all the 5-paths of G are of the type $x_{i_1}\, x_{i_2}\, x_{i_3}\, x_{i_4}\, x_{i_5}\, x_{i_6}$, $1 \leqslant i_1 \neq \neq i_2 \neq i_3 \neq i_4 \neq i_5 \neq i_6 \leqslant p$, where the vertices correspond to the above rows of M_G, and the generators of $I_6(G)$ are the 5-paths of G different from one another for at least an index.

Proof. To construct 5-paths in G, let's start from a 3-path whose internal vertices are given by the rows R_{i_3}, R_{i_4} with two 1's in any $(p{\times}3)$-submatrix B_ℓ of M_G, and whose ends by the rows R_{i_2}, R_{i_5} of B_ℓ with a unique 1.
These four vertices represent the internal vertices of the 5-paths can be obtained from B_ℓ.
By similar reasoning as in the proof of Theorem 1, the ends of the 5-paths in G that come from B_ℓ are given by well-determined rows R_{i_1}, R_{i_6} of M_G, different from each other and from R_{i_2}, R_{i_3}, R_{i_4}, R_{i_5}.

Such 5-paths, for every choice of B_ℓ, have the form $x_{i_1}\,x_{i_2}\,x_{i_3}\,x_{i_4}\,x_{i_5}\,x_{i_6}$.
The last assertion derives from the definition of generalized graph ideal.

Exercise 1 Consider the following graph G having a triangular cycle subgraph



and compute the number of $(q-1)$-paths of G and the generators of the generalized graph ideal $I_q(G)$, $q>2$, using only the incidence matrix of G.

Vertices of G are the generators of $I_1(G) = (\,x_1,\, x_2,\, x_3,\, x_4,\, x_5,\, x_6,\, x_7\,)$.

Edges of G are the generators of
$I_2(G) = (\,x_1\,x_2,\; x_1\,x_3,\; x_2\,x_3,\; x_3\,x_4,\; x_4\,x_5,\; x_4\,x_6,\; x_6\,x_7\,)$.

The incidence matrix of G is $\quad M_G = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$.

Observe that the maximal length of the paths of G is 5;
therefore the generalized graph ideal $I_q(G)$ exists for $q \leqslant 6$.

The number of 2-paths of G is: $\binom{2}{2}+\binom{2}{2}+\binom{3}{2}+\binom{3}{2}+\binom{2}{2} = 9$.

Such 2-paths are: $x_1\,x_2\,x_3,\;\; x_1\,x_3\,x_2,\;\; x_1\,x_3\,x_4,\;\; x_2\,x_1\,x_3,\;\; x_2\,x_3\,x_4,$
$x_3\,x_4\,x_5,\;\; x_3\,x_4\,x_6,\;\; x_4\,x_6\,x_7,\;\; x_5\,x_4\,x_6$.

So $\;I_3(G) = (\,x_1\,x_2\,x_3,\;\; x_1\,x_3\,x_4,\;\; x_2\,x_3\,x_4,\;\; x_3\,x_4\,x_5,\;\; x_3\,x_4\,x_6,$
$x_4\,x_6\,x_7,\;\; x_5\,x_4\,x_6\,)$ is generated by 7 2-paths.

The number of 3-paths of G is: $\begin{bmatrix}2\\2\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}3\\3\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}-3 =$
$= 4+3\cdot 2+1-3 = 8$.

Such 3-paths are: $x_1\,x_2\,x_3\,x_4,\;\; x_1\,x_3\,x_4\,x_5,\;\; x_1\,x_3\,x_4\,x_6,\;\; x_2\,x_1\,x_3\,x_4,$
$x_2\,x_3\,x_4\,x_5,\;\; x_2\,x_3\,x_4\,x_6,\;\; x_3\,x_4\,x_6\,x_7,\;\; x_5\,x_4\,x_6\,x_7$.

So $\;I_4(G) = (\,x_1\,x_2\,x_3\,x_4,\;\; x_1\,x_3\,x_4\,x_5,\;\; x_1\,x_3\,x_4\,x_6,\;\; x_2\,x_3\,x_4\,x_5,\;\; x_2\,x_3\,x_4\,x_6,$
$x_3\,x_4\,x_6\,x_7,\;\; x_5\,x_4\,x_6\,x_7\,)$ is generated by 7 3-paths.

To determine the number of 4-paths of G, consider the $(7{\times}2)$-submatrices of M_G having one row of 1's and other two rows both with a unique 1:

$$\begin{pmatrix}1&1\\1&0\\0&1\\0&0\\0&0\\0&0\\0&0\end{pmatrix},\begin{pmatrix}1&0\\1&1\\0&1\\0&0\\0&0\\0&0\\0&0\end{pmatrix},\begin{pmatrix}1&0\\0&1\\1&1\\0&0\\0&0\\0&0\\0&0\end{pmatrix},\begin{pmatrix}1&0\\0&0\\1&1\\0&1\\0&0\\0&0\\0&0\end{pmatrix},\begin{pmatrix}0&0\\1&0\\1&1\\0&1\\0&0\\0&0\\0&0\end{pmatrix},\begin{pmatrix}0&0\\0&0\\1&0\\1&1\\0&1\\0&0\\0&0\end{pmatrix},\begin{pmatrix}0&0\\0&0\\1&0\\1&1\\0&0\\0&1\\0&0\end{pmatrix},\begin{pmatrix}0&0\\0&0\\0&0\\1&1\\1&0\\0&1\\0&0\end{pmatrix},\begin{pmatrix}0&0\\0&0\\0&0\\1&0\\0&0\\1&1\\0&1\end{pmatrix}.$$

Then it is: $\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}2\\2\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}-(3+2)=5\cdot2+1-5=6\,.$

Such 4-paths are: $x_1\,x_2\,x_3\,x_4\,x_5,\;\; x_1\,x_2\,x_3\,x_4\,x_6,\;\; x_1\,x_3\,x_4\,x_6\,x_7,$
$x_2\,x_1\,x_3\,x_4\,x_5,\;\; x_2\,x_1\,x_3\,x_4\,x_6,\;\; x_2\,x_3\,x_4\,x_6\,x_7\,.$

So $I_5(G)=(\,x_1\,x_2\,x_3\,x_4\,x_5,\;\; x_1\,x_2\,x_3\,x_4\,x_6,\;\; x_1\,x_3\,x_4\,x_6\,x_7,\;\; x_2\,x_3\,x_4\,x_6\,x_7\,)$ is generated by 4 4-paths.

To determine the number of 5-paths of G, consider the $(7{\times}3)$-submatrices of M_G having just two rows both with two 1's and other two rows both with a unique 1:

$$\begin{pmatrix}1&1&0\\1&0&0\\0&1&1\\0&0&1\\0&0&0\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}1&0&0\\1&1&0\\0&1&1\\0&0&1\\0&0&0\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}1&0&0\\0&0&0\\1&1&0\\0&1&1\\0&0&1\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}1&0&0\\0&0&0\\1&1&0\\0&1&1\\0&0&0\\0&0&1\\0&0&0\end{pmatrix},$$

$$\begin{pmatrix}0&0&0\\1&0&0\\1&1&0\\0&1&1\\0&0&1\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\1&0&0\\1&1&0\\0&1&1\\0&0&0\\0&0&1\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\\1&1&0\\0&0&0\\0&1&1\\0&0&1\end{pmatrix},\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\\1&1&0\\1&0&0\\0&1&1\\0&0&1\end{pmatrix}.$$

Then it is: $\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}2\\2\end{bmatrix}+\begin{bmatrix}2\\2\end{bmatrix}-2\cdot2=2\cdot2+2\cdot1-4=2\,.$

Such 5-paths are: $x_1\,x_2\,x_3\,x_4\,x_6\,x_7,\;\; x_2\,x_1\,x_3\,x_4\,x_6\,x_7\,.$

So $I_6(G)=(\,x_1\,x_2\,x_3\,x_4\,x_6\,x_7\,)$ is generated by one 5-path.

Remark 3 For cycles, complete graphs and complete bipartite graphs, we are able to determine the number of paths of any length, but their composition is not completely studied for lengths greater than 5 . In fact :

1. <u>Cycles</u> $C_n\,,\ n\geqslant 3$.

Vertices in C_n have degree 2 ; the incidence matrix M_{C_n} is an $(n{\times}n)$-matrix.

C_n has n edges (or 1-paths);
C_n has n 2-paths (n, number of the rows of M_{C_n});
C_n has n 3-paths (n, number of the columns of M_{C_n});
....
C_n has n $(n-1)$-paths.

When $3 \leqslant q \leqslant 6$, Props 1, 2 and Thms 1, 2 give the structure of $(q–1)$-paths of C_n and the generators of the generalized graph ideals $I_q(C_n)$.

2. Complete graphs K_n, $n \geqslant 3$.
Vertices in K_n have degree $n-1$; there are $\frac{(k-1)!}{2}\binom{n}{k}$ cycles C_k, $k \geqslant 3$; the incidence matrix M_{K_n} is an $\left(n \times \frac{n(n-1)}{2}\right)$-matrix.

K_n has $\binom{n}{2} = \frac{n(n-1)}{2}$ edges (or 1-paths);

K_n has $n\binom{n-1}{2} = \frac{n(n-1)(n-2)}{2}$ 2-paths;

K_n has $\binom{n}{2}\begin{bmatrix} n-1 \\ n-1 \end{bmatrix} - 3\binom{n}{3} = \frac{n(n-1)(n-2)(n-3)}{2}$ 3-paths;

$$K_n \text{ has } n\binom{n-1}{2}\begin{bmatrix} n-1 \\ n-1 \end{bmatrix} - (3 + 2\cdot 3\,(n-3))\binom{n}{3} - 4\cdot 3\binom{n}{4} = \\ = \frac{n(n-1)(n-2)(n-3)(n-4)}{2} \quad \text{4-paths};$$

$$K_n \text{ has } \frac{n(n-1)(n-2)(n-3)}{2}\begin{bmatrix} n-1 \\ n-1 \end{bmatrix} - 2\cdot 3\,(n-2)\,(n-3)\binom{n}{3} + \\ + 2\binom{n}{2}\binom{n-2}{2} - 3\,(4 + 2\cdot 4\,(n-4))\binom{n}{4} - 5\cdot 12\binom{n}{5} = \\ = \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{2} \quad \text{5-paths};$$

....

K_n has $\frac{n(n-1)(n-2)\cdots 2\cdot 1}{2} = \frac{n!}{2}$ $(n-1)$-paths.

When $3 \leqslant q \leqslant 6$, Props 1, 2 and Thms 1, 2 give the structure of $(q–1)$-paths of K_n and the generators of the generalized graph ideals $I_q(K_n)$.

3. Complete bipartite graphs $K_{m,n}$, $m+n \geqslant 3$.
m vertices of $K_{m,n}$ have degree n; n vertices have degree m;
in $K_{m,n}$ there are $\frac{k!\,k!}{2k}\binom{m}{k}\binom{n}{k}$ cycles C_{2k}, $k \geqslant 2$;
the incidence matrix $M_{K_{m,n}}$ is an $\big((m+n)\times m\,n\big)$-matrix.

$K_{m,n}$ has $m\,n$ edges (or 1-paths);

$K_{m,n}$ has $m\binom{n}{2}+n\binom{m}{2}$ 2-paths;

$K_{m,n}$ has $mn\begin{bmatrix} m\\ n\end{bmatrix}=m\,(m-1)\,n\,(n-1)=2\binom{m}{2}2\binom{n}{2}$ 3-paths;

$K_{m,n}$ has $m\binom{n}{2}\begin{bmatrix} m\\ m\end{bmatrix}+n\binom{m}{2}\begin{bmatrix} n\\ n\end{bmatrix}-4\,\frac{2!\,2!}{4}\binom{m}{2}\binom{n}{2}=$

$$=2\binom{m}{2}2\binom{n}{2}\frac{m-1}{2}+2\binom{m}{2}2\binom{n}{2}\frac{n-1}{2}-2\binom{m}{2}2\binom{n}{2}=$$
$$=m\,(m-1)\,n\,(n-1)\,\frac{(m-2)+(n-2)}{2}=$$
$$=3!\binom{m}{3}\binom{n}{2}+\binom{m}{2}3!\binom{n}{3}\quad \text{4-paths;}$$

$K_{m,n}$ has $mn\begin{bmatrix} m\\ n\end{bmatrix}\begin{bmatrix} m\\ n\end{bmatrix}-(4+2\,(2\,(m-2)+2\,(n-2)))\,\frac{2!\,2!}{4}\binom{m}{2}\binom{n}{2}=$

$$=2\binom{m}{2}2\binom{n}{2}\left((m-1)\,(n-1)-(1+(m-2)+(n-2))\right)=$$
$$=m\,(m-1)\,n\,(n-1)\;\left(mn-m-n+1-m-n+3\right)=$$
$$=m\,(m–1)\,(m–2)\,n\,(n–1)\,(n–2)=3!\binom{m}{3}+3!\binom{n}{3}\quad \text{5-paths;}$$

....

$K_{m,n}$ has $\frac{h!}{2}\binom{m}{h}(h+1)!\binom{n}{h+1}+(h+1)!\binom{m}{h+1}\frac{h!}{2}\binom{n}{h}$ $(m+n–1)$-paths

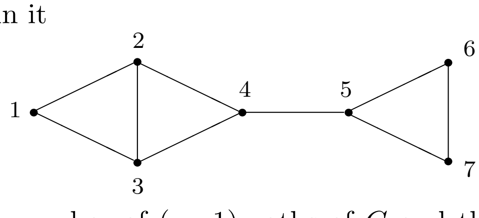
if $m+n=2\,h+1$ is odd,

$h!\binom{m}{h}h!\binom{n}{h}$ $(m+n-1)$-paths if $m+n=2\,h$ is even.

(see [3], Corollary 4).

When $3\leqslant q\leqslant 6$, Props 1, 2 and Thms 1, 2 give the structure of $(q–1)$-paths of $K_{m,n}$ and the generators of the generalized graph ideals $L_q(K_{m,n})$.

Exercise 2 Consider the following graph G with triangular and squared cycle subgraphs in it



and compute the number of $(q-1)$-paths of G and the generators of the generalized graph ideal $I_q(G)$, $q>2$, using only the incidence matrix of G.

Vertices of G are the generators of $I_1(G)=(x_1,\,x_2,\,x_3,\,x_4,\,x_5,\,x_6,\,x_7)$.

Edges of G are the generators of

$I_2(G)=(x_1x_2,\,x_1x_3,\,x_2x_3,\,x_2x_4,\,x_3x_4,\,x_4x_5,\,x_5x_6,\,x_5x_7,\,x_6x_7)$.

The incidence matrix of G is $M_G = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$.

Observe that the maximal length of the paths of G is 6;
therefore the generalized graph ideal $I_q(G)$ exists for $q \leqslant 7$.

The number of 2-paths of G is:

$$\binom{2}{2}+\binom{3}{2}+\binom{3}{2}+\binom{3}{2}+\binom{3}{2}+\binom{2}{2}+\binom{2}{2} = 15\,.$$

Such 2-paths are: $x_1\,x_2\,x_3,\ \ x_1\,x_3\,x_2,\ \ x_1\,x_2\,x_4,\ \ x_1\,x_3\,x_4,\ \ x_2\,x_1\,x_3,$
$x_2\,x_3\,x_4,\ \ x_2\,x_4\,x_3,\ \ x_2\,x_4\,x_5,\ \ x_3\,x_2\,x_4,\ \ x_3\,x_4\,x_5,$
$x_4\,x_5\,x_6,\ \ x_4\,x_5\,x_7,\ \ x_5\,x_6\,x_7,\ \ x_5\,x_7\,x_6,\ \ x_6\,x_5\,x_7\,.$

So $I_3(G) = (\,x_1\,x_2\,x_3,\ \ x_1\,x_2\,x_4,\ \ x_1\,x_3\,x_4,\ \ x_2\,x_3\,x_4,\ \ x_2\,x_4\,x_5,\ \ x_3\,x_4\,x_5,$
$x_4\,x_5\,x_6,\ \ x_4\,x_5\,x_7,\ \ x_5\,x_6\,x_7\,)$ is generated by 9 2-paths.

The number of 3-paths is:

$$\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}3\\3\end{bmatrix}+\begin{bmatrix}3\\3\end{bmatrix}+\begin{bmatrix}3\\3\end{bmatrix}+\begin{bmatrix}3\\3\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}2\\2\end{bmatrix}-3\cdot 3 =$$
$$= 4\begin{bmatrix}3\\3\end{bmatrix}+4\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}2\\2\end{bmatrix}-9 = 16+8+1-9 = 16\,.$$

Such 3-paths are: $x_1\,x_2\,x_3\,x_4,\ \ x_1\,x_2\,x_4\,x_3,\ \ x_1\,x_2\,x_4\,x_5,\ \ x_1\,x_3\,x_2\,x_4,$
$x_1\,x_3\,x_4\,x_2,\ \ x_1\,x_3\,x_4\,x_5,\ \ x_2\,x_1\,x_3\,x_4,\ \ x_2\,x_3\,x_4\,x_5,$
$x_2\,x_4\,x_5\,x_6,\ \ x_2\,x_4\,x_5\,x_7,\ \ x_3\,x_1\,x_2\,x_4,\ \ x_3\,x_2\,x_4\,x_5,$
$x_3\,x_4\,x_5\,x_6,\ \ x_3\,x_4\,x_5\,x_7,\ \ x_4\,x_5\,x_6\,x_7,\ \ x_4\,x_5\,x_7\,x_6\,.$

So $I_4(G) = (\,x_1\,x_2\,x_3\,x_4,\ \ x_1\,x_2\,x_4\,x_5,\ \ x_1\,x_3\,x_4\,x_5,\ \ x_2\,x_3\,x_4\,x_5,\ \ x_2\,x_4\,x_5\,x_6,$
$x_2\,x_4\,x_5\,x_7,\ \ x_3\,x_4\,x_5\,x_6,\ \ x_3\,x_4\,x_5\,x_7,\ \ x_4\,x_5\,x_6\,x_7\,)$
is generated by 9 3-paths.

To determine the number of 4-paths of G, consider the $(7{\times}2)$-submatrices of M_G having one row of 1's and other two rows both with a unique 1:

$$\begin{pmatrix}1&1\\1&0\\0&1\\0&0\\0&0\\0&0\\0&0\end{pmatrix},\ \begin{pmatrix}1&0\\1&1\\0&1\\0&0\\0&0\\0&0\\0&0\end{pmatrix},\ \begin{pmatrix}1&0\\1&1\\0&0\\0&1\\0&0\\0&0\\0&0\end{pmatrix},\ \begin{pmatrix}1&0\\0&1\\1&1\\0&0\\0&0\\0&0\\0&0\end{pmatrix},\ \begin{pmatrix}1&0\\0&0\\1&1\\0&1\\0&0\\0&0\\0&0\end{pmatrix},\ \begin{pmatrix}0&0\\1&1\\1&0\\0&1\\0&0\\0&0\\0&0\end{pmatrix},\ \begin{pmatrix}0&0\\1&0\\1&1\\0&1\\0&0\\0&0\\0&0\end{pmatrix},\ \begin{pmatrix}0&0\\1&0\\0&1\\1&1\\0&0\\0&0\\0&0\end{pmatrix},$$

$$\begin{pmatrix}0&0\\1&0\\0&0\\1&1\\0&1\\0&0\\0&0\end{pmatrix},\begin{pmatrix}0&0\\0&0\\1&0\\1&1\\0&1\\0&0\\0&0\end{pmatrix},\begin{pmatrix}0&0\\0&0\\0&0\\1&0\\1&1\\0&1\\0&0\end{pmatrix},\begin{pmatrix}0&0\\0&0\\0&0\\1&0\\1&1\\0&0\\0&1\end{pmatrix},\begin{pmatrix}0&0\\0&0\\0&0\\0&0\\1&1\\1&0\\0&1\end{pmatrix},\begin{pmatrix}0&0\\0&0\\0&0\\0&0\\1&0\\1&1\\0&1\end{pmatrix},\begin{pmatrix}0&0\\1&0\\0&1\\0&0\\1&0\\0&1\\1&1\end{pmatrix}.$$

Then it is: $\begin{bmatrix}3\\3\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}3\\3\end{bmatrix}+\begin{bmatrix}3\\3\end{bmatrix}+\begin{bmatrix}3\\3\end{bmatrix}+\begin{bmatrix}3\\3\end{bmatrix}+\begin{bmatrix}3\\3\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}+$

$$+\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}2\\2\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}-(3+4)-(3+6)-(3+2)-4=$$

$$=6\begin{bmatrix}3\\3\end{bmatrix}+8\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}2\\2\end{bmatrix}-7-9-5-4=24+16+1-25=16\,.$$

Such 4-paths are:

$$x_1\,x_2\,x_3\,x_4\,x_5,\;\; x_1\,x_2\,x_4\,x_5\,x_6,\;\; x_1\,x_2\,x_4\,x_5\,x_7,\;\; x_1\,x_3\,x_2\,x_4\,x_5,$$
$$x_1\,x_3\,x_4\,x_5\,x_6,\;\; x_1\,x_3\,x_4\,x_5\,x_7,\;\; x_2\,x_1\,x_3\,x_4\,x_5,\;\; x_2\,x_3\,x_4\,x_5\,x_6,$$
$$x_2\,x_3\,x_4\,x_5\,x_7,\;\; x_2\,x_4\,x_5\,x_6\,x_7,\;\; x_2\,x_4\,x_5\,x_7\,x_6,\;\; x_3\,x_1\,x_2\,x_4\,x_5,$$
$$x_3\,x_2\,x_4\,x_5\,x_6,\;\; x_3\,x_2\,x_4\,x_5\,x_7,\;\; x_3\,x_4\,x_5\,x_6\,x_7,\;\; x_3\,x_4\,x_5\,x_7\,x_6\,.$$

So $I_5(G)=(\,x_1\,x_2\,x_3\,x_4\,x_5,\;\; x_1\,x_2\,x_4\,x_5\,x_6,\;\; x_1\,x_2\,x_4\,x_5\,x_7,$
$x_1\,x_3\,x_4\,x_5\,x_6,\;\; x_1\,x_3\,x_4\,x_5\,x_7,\;\; x_2\,x_3\,x_4\,x_5\,x_6,$
$x_2\,x_3\,x_4\,x_5\,x_7,\;\; x_2\,x_4\,x_5\,x_6\,x_7,\;\; x_3\,x_4\,x_5\,x_6\,x_7\,)$
is generated by 9 4-paths.

To determine the number of 5-paths of G, consider the $(7{\times}3)$-submatrices of M_G having just two rows both with two 1's and other two rows both with a unique 1:

$$\begin{pmatrix}1&1&0\\1&0&1\\0&1&0\\0&0&1\\0&0&0\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}1&1&0\\1&0&0\\0&1&1\\0&0&1\\0&0&0\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}1&0&0\\1&1&0\\0&1&1\\0&0&1\\0&0&0\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}1&0&0\\1&1&0\\0&0&1\\0&1&1\\0&0&0\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}1&0&0\\1&1&0\\0&0&0\\0&1&1\\0&0&1\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}1&0&0\\0&1&1\\1&1&0\\0&0&1\\0&0&0\\0&0&0\\0&0&0\end{pmatrix},$$

$$\begin{pmatrix}1&0&0\\0&1&0\\1&0&1\\0&1&1\\0&0&0\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}1&0&0\\0&0&0\\1&1&0\\0&1&1\\0&0&1\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\1&1&0\\1&0&0\\0&1&1\\0&0&1\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\1&0&0\\1&1&0\\0&1&1\\0&0&1\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\\1&1&0\\0&1&1\\0&0&1\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\\1&1&0\\0&1&1\\0&0&0\\0&0&1\end{pmatrix},$$

$$\begin{pmatrix} 0&0&0\\0&0&0\\1&0&0\\1&1&0\\0&1&1\\0&0&1\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&0\\1&0&0\\1&1&0\\0&1&1\\0&0&0\\0&0&1 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0\\1&0&0\\1&1&0\\0&1&1\\0&0&1 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0\\1&0&0\\1&1&0\\0&0&1\\0&1&1 \end{pmatrix}.$$

Then it is:

$$\begin{bmatrix}3\\3\end{bmatrix}+\begin{bmatrix}3\\3\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}3\\3\end{bmatrix}+\begin{bmatrix}3\\3\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}+$$
$$+\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}3\\2\end{bmatrix}-2\,(\,(2+2)+(1+1+2)+2\,)+2\cdot 1-(4+6-2\cdot 2)=$$
$$=4\begin{bmatrix}3\\3\end{bmatrix}+12\begin{bmatrix}3\\2\end{bmatrix}-2\cdot 10+2-6=16+24-24=16\,.$$

Such 5-paths are:

$$\begin{array}{llll} x_1\,x_2\,x_3\,x_4\,x_5\,x_6, & x_1\,x_2\,x_3\,x_4\,x_5\,x_7, & x_1\,x_2\,x_4\,x_5\,x_6\,x_7, & x_1\,x_2\,x_4\,x_5\,x_7\,x_6,\\ x_1\,x_3\,x_2\,x_4\,x_5\,x_6, & x_1\,x_3\,x_2\,x_4\,x_5\,x_7, & x_1\,x_3\,x_4\,x_5\,x_6\,x_7, & x_1\,x_3\,x_4\,x_5\,x_7\,x_6,\\ x_2\,x_1\,x_3\,x_4\,x_5\,x_6, & x_2\,x_1\,x_3\,x_4\,x_5\,x_7, & x_2\,x_3\,x_4\,x_5\,x_6\,x_7, & x_2\,x_3\,x_4\,x_5\,x_7\,x_6,\\ x_3\,x_1\,x_2\,x_4\,x_5\,x_6, & x_3\,x_1\,x_2\,x_4\,x_5\,x_7, & x_3\,x_2\,x_4\,x_5\,x_6\,x_7, & x_3\,x_2\,x_4\,x_5\,x_7\,x_6\,. \end{array}$$

So $I_6(G) = (\,x_1\,x_2\,x_3\,x_4\,x_5\,x_6,\ \ x_1\,x_2\,x_3\,x_4\,x_5\,x_7,\ \ x_1\,x_2\,x_4\,x_5\,x_6\,x_7,$
$x_1\,x_3\,x_4\,x_5\,x_6\,x_7,\ \ x_2\,x_3\,x_4\,x_5\,x_6\,x_7\,)$
is generated by 5 5-paths.

It remains to determine the number and the composition of 6-paths of G; in fact we noticed that in G there not exist paths of length greater than 6. We cannot still able to do this, but we can only say that the generalized graph ideal $I_7(G)$ is generated by one 6-path, evidently $x_1\,x_2\,x_3\,x_4\,x_5\,x_6\,x_7$.

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CLASSES OF SAMPLES AND TOTAL ORDERS

VINCENZO IORFIDA - GAETANA RESTUCCIA

Abstract. For bernoullian samples one introduces total orders to obtain slender procedures in the finding critical regions for statistical tests for media.

Classification AMS: 13Pxx, 62-xx, 62D05.

Introduction

Statistical methods are applied to answer questions in many of the major business disciplines including accounting, finance, marketing, economy, production and in general, management. The aim of this paper is to introduce total orders on the semigroup $\mathbb{N}^n$, as the degreelexicographic order or the reversedegree lexicographic order, in order to obtain subsets of $\mathbb{N}^n$ totally ordered. If we consider a bernoullian sample, the data of this sample are a subset of $\mathbb{N}^n$, then in this case we can order the sample data. This is a good result that can be utilized when we have the problem to find the critical region for an hypothesis test which involves mean or variance.
In N.1 we give some definitions about samples and we recall the definition of four total orders in $\mathbb{N}^n$ and the compatibility of a statistic with respect to the total order introduced. In N.2 we apply these concepts to find the critical regions for an hypothesis test for mean or variance of a bernoullian sample.

1. Total orders

We recall some definitions.

Definition 1.1. *The total group of objects being studied or investigated is called population.*

Definition 1.2. *A group of objects that is selected from population and from which information is gathered is called sample.*

Definition 1.3. *The methods involved in drawing conclusions about population based upon information collected from a sample constitute generally a statistical domain, said statistical inference.*

Algebraic methods can be used where algebraic functions are utilized in our process. On the other and, algebraic functions are very important:

1) $\overline{X}$: the sample mean;

2) σ^2: the sample variance.

$\overline{X} = \frac{1}{n}(X_1 + X_2 + \cdots + X_n)$, where n is the sample size and $X_1, X_2, \ldots, X_n$ are the random variables of the random process. Hence $\overline{X}$ is a linear function of n variables $X_1, X_2, \ldots, X_n$, i.e. a linear polynomial of $\mathbb{R}[X_1, X_2, \ldots, X_n]$, $\mathbb{R}$ the field of real numbers. Moreover $\sigma^2 = \frac{1}{n}[(X_1 - \mu)^2 + \cdots + (X_n - \mu)^2]$, where μ is the population mean (value being estimate) is a polynomial of $\mathbb{R}[X_1, X_2, \ldots, X_n]$ of degree two. We call $\overline{X}$ and σ^2 statistics.
In general we can have other statistics in our process, but these are the most studied and important. Methods of computational algebra can help us to study interesting questions.

Definition 1.4. *Let $X_1, X_2, \ldots, X_n$ be random variables and $\underline{X} = (X_1, X_2, \ldots, X_n)$ be the correspondent sample. Let $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ be two sample data. Let $(a_1, \ldots, a_n) < (b_1, \ldots, b_n)$ if $(a_1, \ldots, a_n) < (b_1, \ldots, b_n)$ for a total order in $\mathbb{R}^n$.*

As a consequence we can order all sample data of the sample X. From now on , the space of sample data will be totally ordered.

Definition 1.5. *Let $s(X_1, X_2, \ldots, X_n) \in \mathbb{R}[X_1, X_2, \ldots, X_n]$ a statistic. An order on the random variables $X_1, X_2, \ldots, X_n$ is called compatible with the statistic function $s(X_1, X_2, \ldots, X_n)$ if $(a_1, \ldots, a_n) < (b_1, \ldots, , b_n)$ implies $s(a_1, \ldots, a_n) \leq s(b_1, \ldots, , b_n)$ for every sample data.*

2. Bernoullian samples

Let $\mathbb{N}$ be the set of natural numbers. We define a total order in the semigroup $\mathbb{N}^n$ as follows:

1. The lexicographical order: for $\mathbf{a} = (a_1, \ldots, a_n)$, $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{N}^n$ we define $x^a < x^b$ if and only if the first coordinates a_i and b_i in $\mathbf{a}$ and $\mathbf{b}$ from the left, which are different, satisfy $a_i < b_i$.

2. The reverse lexicographical order: for $\mathbf{a} = (a_1, \ldots, a_n)$, $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{N}^n$ we define $x^a < x^b$ if and only if the first coordinates a_i and b_i in $\mathbf{a}$ and $\mathbf{b}$ from the left, which are different, satisfy $a_i > b_i$.

3. The degree lexicographical order: For $\mathbf{a} = (a_1, \ldots, a_n)$, $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{N}^n$ we define $x^a < x^b$ if and only if

$\sum_{i=1}^n a_i < \sum_{i=1}^n b_i$ or

$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i$ and $x^a < x^b$ with respect to lex order with $x_1 > x_2 > \cdots > x_n$.

4. The degree reverse lexicographical order: for $\mathbf{a}= (a_1, \ldots, a_n)$, $\mathbf{b}= (b_1, \ldots, b_n) \in \mathbb{N}^n$ we define $x^a < x^b$ if and only if

   $\sum_{i=1}^{n} a_i < \sum_{i=1}^{n} b_i$ or

   $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i$ and the first coordinates a_1 and b_i in **a** e **b** from the right, which are different, satisfy $a_i > b_i$.

Example 2.1. *The sample mean X is compatible with respect to the degree lexicographic order and not with respect to the lexicographic order: $(1, 4, 5) < (2, 2, 3)$ for lex, but $1 + 4 + 5 > 2 + 2 + 3$.*

Example 2.2. *The sample variance σ^2 is not compatible with respect to the degree lexicographic order, neither with respect to the lexicographic order: $(1, 2, 3) < (2, 4, 5)$ for degree lex order and for lex order, but in both cases, if the population mean $\mu = 3$, $(1-3)^2 + (2-3)^2 + (3-3)^2 > (2-3)^2 + (4-3)^2 + (5-3)^2$.*

We recall the:

Definition 2.3. *A sample $(X_1, \ldots, X_n)$ will said bernoullian if its data are 0 or 1.*

Proposition 2.4. *If $(X_1, \ldots, X_n)$ is a bernoullian sample. The sample mean and the sample variance are compatible with the degree lexicographic order an the reverse degree lexicographic order of $\mathbb{N}^n$.*

Proof:It is sufficient to observe that in the expression of the sample mean and of the sample variance we have the same number of 1 and 0. Hence the assertion.

As an application of the previous concept we consider hypothesis tests on means or variances for bernoullian samples. One of the most common types of hypothesis tests deals with the testing of means.

Example 2.5. *State the null hypothesis (H_0), which is the statement that we test.*
$(H_0) : \mu = a$, $\mu > a$, $\mu < a$, that is the population mean is equal to some specified value or least some specified value of a, or at most some specified value of a. Since the sample mean is compatible with the order degreelex, this implies the rejection region of (H_0) or the fail to reject region of (H_0) can be easily computed accordingly our order $<$.

Theorem 2.6. *Let T be any hypothesis test involving means or variances and suppose that the null hypothesis (H_0) is the statement that we test. Consider the cases*

1) $(H_0) : \mu = a$, *i.e. the population mean is equal to some specified value.*
2) $(H_0) : \sigma^2 = a$, *i.e. the population variance is equal to some specified value.*

Suppose that the fail to reject region A *of* T *is given in terms of inequality on the statistic sample mean* $\overline{X}$,

$$A(\alpha) = \{(a_1, \ldots, a_n) \in \mathbb{N}^n / \overline{X} < c\},$$

or on the statistic sample variance σ^2,

$$A(\alpha) = \{(a_1, \ldots, a_n) \in \mathbb{N}^n / \sigma^2 < c\},$$

where α *is the level of* T *and* $c \in \mathbb{R}$. *Suppose that we have a sample data* $(b_1, \ldots, b_n)$ *such that*

1) $b_1 + \cdots + b_n < cn$ *or*
2) $(b_1 - \mu)^2 + \cdots + (b_n - \mu)^2 < cn$

Then all $(a_1, \ldots, a_n) \in \mathbb{N}^n$ *and such that* $(a_1, \ldots, a_n) < (b_1, \ldots, b_n)$ *for degree lex order or degree revlex order belong to* $A(\alpha)$.

Proof 1) The assertion depends from the compatibility of the statistic sample mean and sample variance (prop. 2.4) with the degree lex order or degree revlex order, hence $(a_1, \ldots, a_n) < (b_1, \ldots, b_n)$ implies $a_1 + \cdots + a_n \leq b_1 + \cdots + b_n < cn$ or $(a_1 - \mu)^2 + \ldots + (a_n - \mu)^2 \leq (b_1 - \mu)^2 + \ldots + (b_n - \mu)^2$
2) Same remarks, since the sample is bernoullian.

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On Betti numbers of a class of bipartite planar graphs

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Abstract: Let B_{2t} be a class of bipartite planar graphs with $t \geq 1$ an integer and $r = 2t$ be the number of its regions.
We study the Betti numbers of the edge ideal of B_{2t} for all $t \geq 1$. We give bounds for the graded Betti numbers and the projective dimension linked to r.
Classification AMS: 13F20, 13C10, 13D02, 05C99.

Introduction

Let G be a graph on a vertex set $V = \{v_1, \ldots, v_n\}$ and $R = K[X_1, \ldots, X_n]$ be the polynomial ring over a field K, with one variable X_i for each vertex v_i of V.
The ideal I_G of R generated by the set $\{X_iX_j | \ \{v_i, v_j\} \ is \ an \ edge \ of \ G\}$ is said the edge ideal of G. In [7] we find some results about monomial ideals of R that can arise from the edges of a graph G.
In this paper we are interested to extract specific informations about some invariants of I_G linked to its resolution when G is a bipartite planar graph by studying the particular geometry of G. A graph is said planar if it is embedded in the plane such that each pair of edges is intersected alone in the common vertices. As a consequence it is divided in some planar regions by its edges. More precisely, we consider the class of bipartite planar graphs B_{2t}, where $t \geq 1$ is an integer and $r = 2t$ is the number of its regions. These graphs have been introduced in [2], where the authors study the K-algebra $K[B_{2t}]$ using the geometry of the graph B_{2t}. In particular they compute the Hilbert function of $K[B_{2t}]$ and bound the coefficients.

In section 1 of this paper we study the second Betti number of degree three of $R/\mathcal{I}$, where $\mathcal{I}$ is the edge ideal of B_{2t}.
In [4] Eliahou and Villarreal find an expression for the second graded Betti number in degree three linked to linear syzygies of the edge ideal in terms of graph properties using the number of triangles of a generic graph G.
Now we link the second graded Betti number in degree three to the number of the regions of these planar graphs and we give an explicit formula to compute it.

In the second part of this paper we study all the graded Betti numbers that appear in the minimal graded resolution of $R/\mathcal{I}$ using some geometric properties of B_{2t}. The graded Betti numbers determine the rank of the free modules appearing in the minimal graded resolution of $R/\mathcal{I}$ and in general it is not possible to give a generic formula to compute them. However we are able to give bounds for them in terms of the number of the regions of B_{2t}. Furthermore, we compute the dimension of $R/\mathcal{I}$ and we establish bounds for the projective dimension of $R/\mathcal{I}$ studying their geometry in the plane in connection to graph theoretical properties.

The author is grateful to Professor Gaetana Restuccia for useful discussions about the results of this paper.

1 The second graded Betti number of B_{2t}

Let G be a graph with a finite vertex set $V = \{v_1, \ldots, v_n\}$ and edge set E, that consists of pairs $\{v_i, v_j\}$ said edges, for some $v_i, v_j \in V$.
Let $R = K[X_1, \ldots, X_n]$ be the polynomial ring over a field K, with one variable X_i for each vertex v_i of G.

The *edge ideal* I_G associated to a graph G is the ideal of R generated by monomials of degree two, X_iX_j, on the variables $X_1, \ldots, X_n$, such that $\{v_i, v_j\} \in E$ for $1 \le i, j \le n$:

$$I_G = (\{X_iX_j | \{v_i, v_j\} \in E\}).$$

A graph G on vertices $v_1, \ldots, v_n$ is *complete* if there exists an edge for all pair $\{v_i, v_j\}$ of vertices of G. It is denoted K_n.

A graph G is *bipartite* if its vertex set V can be partitioned into disjoint subsets $V_1 = \{x_1, \ldots, x_n\}$ and $V_2 = \{y_1, \ldots, y_m\}$, and any edge joins a vertex of V_1 to a vertex of V_2.
A bipartite graph G is *complete* if all the vertices of V_1 are joined to all the vertices of V_2. It is denoted by $K_{n,m}$.

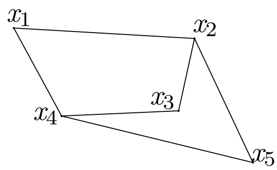
Bipartite graphs determine monomial ideals in the polynomial ring in two sets of variables $R = K[X_1, \ldots, X_n; Y_1, \ldots, Y_m]$, where n is the number of the vertices $x_1, \ldots, x_n$ and m is the number of the vertices $y_1, \ldots, y_m$.
The edge ideal I_G associated to a bipartite graph G is the ideal of R that is generated by the monomials of degree two, X_iY_j, on the variables $X_1, \ldots, X_n; Y_1, \ldots, Y_m$, such that $\{x_i, y_j\} \in E$ for $1 \le i \le n$ and $1 \le j \le m$:

$$I_G = (\{X_iY_j | \{x_i, y_j\} \in E\}).$$

Definition 1.1 *A graph G is planar if it has an embedding in the plane such that each pair of edges is intersected alone in the common vertices.*

Remark 1.1 *A planar graph is divided by its edges in plane regions.*

Example 1.1 *The following picture is a planar graph divided by its edges in 2 regions*



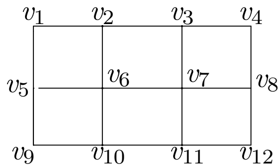
Remark 1.2 *The complete graphs K_5 and $K_{3,3}$ are the minimal not planar graphs. In fact it is not possible to represent these graphs in the plane so that the edges are not intersected alone in the vertices.*

Theorem 1.1 *([5],Th. 11.13)*
A graph is planar if and only if it has no subgraphs containing K_5 and $K_{3,3}$.

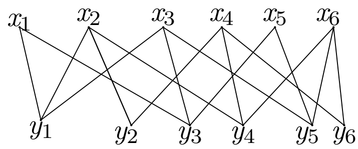
Now we consider a class of bipartite planar graphs introduced in [2]. Let B_{2t} be the planar graph with $r = 2t$ regions, $t \geq 1$ an integer, on vertex set $V(B_{2t}) = \{v_1, \ldots, v_{3t+3}\}$ and edge set $E(B_{2t}) = \{\{v_i, v_{i+1}\} | 1 \leq i \leq 3t+2, \ \ i \neq t+1, t+2\} \cup \{\{v_i, v_{i+t+1}\} | 1 \leq i \leq 2t+2\}$.
B_{2t} is a planar graph by Theorem 1.1, for all $t \geq 1$.

Remark 1.3 *B_{2t} is a bipartite planar graph. The vertex set of B_{2t} can be partitioned into disjoint subsets V_1 and V_2, with $|V_1|+|V_2| = 3t+3$ and $|V_i|$ denotes the number of vertices of V_i.*
We have two cases:
1) t even (and $N = 3t+3$ odd)
$V_1 = \{v_i | i \ \ odd, 1 \leq i \leq 3t+3\}$ with $|V_1| = \frac{3t+4}{2}$
$V_2 = \{v_i | i \ \ even, 1 \leq i \leq 3t+3\}$ with $|V_2| = \frac{3t+2}{2}$
2) t odd (and $N = 3t+3$ even)
$V_1 = \{v_1, v_3, \ldots, v_t\} \cup \{v_{2+(t+1)}, v_{4+(t+1)}, \ldots, v_{t+1+(t+1)}\} \cup \{v_{1+(2t+2)}, v_{3+(2t+2)}, \ldots, v_{t+(2t+2)}\}$ and $V_2 = \{v_2, v_4, \ldots, v_{t+1}\} \cup \{v_{1+(t+1)}, v_{3+(t+1)}, \ldots, v_{t+(t+1)}\} \cup \{v_{2+(2t+2)}, v_{4+(2t+2)}, \ldots, v_{t+1+(2t+2)}\}$.
Note that $|\{v_1, v_3, v_5, \ldots, v_t\}| = |\{v_2, v_4, v_6, \ldots, v_{t+1}\}| = \frac{t+1}{2}$, hence one has $|V_1| = |V_2| = \frac{3t+3}{2}$.
Then the graph B_{2t} has vertex set $V(B_{2t}) = V_1 \cup V_2$ with $V_1 \cap V_2 = \emptyset$ such that its edges join the vertices of V_1 only to vertices of V_2 as follows by definition of $E(B_{2t})$.

Example 1.2 *$G = B_6$, with $V(B_6) = \{v_1, \ldots, v_{12}\}$ and*
$E(B_6) = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_5, v_6\}, \{v_6, v_7\}, \{v_7, v_8\}, \{v_9, v_{10}\}, \{v_{10}, v_{11}\}, \{v_{11}, v_{12}\}, \{v_1, v_5\}, \{v_2, v_6\}, \{v_3, v_7\}, \{v_4, v_8\}, \{v_5, v_9\}, \{v_6, v_{10}\}, \{v_7, v_{11}\}, \{v_8, v_{12}\}\}$



$V(B_6)$ can be partitioned into disjoint subsets:
$V(B_6) = \{v_1, v_3, v_6, v_8, v_9, v_{11}\} \cup \{v_2, v_4, v_5, v_7, v_{10}, v_{12}\} = V_1 \cup V_2$.
If we recall $\{x_1, \ldots, x_6\}$ the vertices of V_1 and $\{y_1, \ldots, y_6\}$ the vertices of V_2 , then the edge set can be written:
$E(B_6) = \{\{x_1, y_1\}, \{x_2, y_1\}, \{x_2, y_2\}, \{x_3, y_3\}, \{x_4, y_4\}, \{x_5, y_5\}, \{x_6, y_5\}, \{x_6, y_6\}, \{x_1, y_3\}, \{x_3, y_1\}, \{x_2, y_4\}, \{x_4, y_2\}, \{x_5, y_3\}, \{x_3, y_5\}, \{x_6, y_4\}, \{x_4, y_6\}, \{x_3, y_4\}\}$



The two pictures represent the same planar graph B_6.

Remark 1.4 *Let $G = B_{2t}$ on vertex set $V(B_{2t}) = V_1 \cup V_2$. The edge ideal $\mathcal{I}$ of B_{2t} that arises from the edge set $E(B_{2t})$ is an ideal of $R = K[X_1, \cdots, X_n; Y_1, \cdots, Y_m]$ where $n = |V_1|$ and $m = |V_2|$.*

Let G be a graph and $I_G \subset R$ its edge ideal. An interesting problem is to express the second graded Betti number in degree three of R/I_G in terms of graph theoretical properties. It is the number of the minimal linear syzygies that generate the first syzygy module of I_G. Eliahou and Villarreal give an explicit formula to compute it ([4]).

Theorem 1.2 *([4], Proposition 2.1)*
Let G be a graph and I_G be the edge ideal. If

$$\ldots \to R^c(-4) \oplus R^b(-3) \to R^q(-2) \to R \to R/I_G \to 0$$

is the minimal graded resolution of R/I_G and $L(G)$ is the edge graph of G, then

$$b = |E(L(G))| - N_3,$$

where N_3 is the number of the triangles of G.

Remark 1.5 *([7],* 6.6.2*)*
The edge graph of G, denoted by $L(G)$, has vertex set equal to the edge set of G and two vertices of $L(G)$ are adjacent whenever the corresponding edges of G have one common vertex:

$$V(L(G)) = E(G) = \{f_1, \ldots, f_q\}$$

$$E(L(G)) = \{(f_i, f_j)|f_i = \{v_i, v_j\}, \quad f_j = \{v_j, v_k\}, \ \imath \neq j, \quad \jmath \neq k\}$$

If G is a graph on vertices $v_1, \ldots, v_n$, then the number of edges of $L(G)$ is given by

$$|E(L(G))| = -|E(G)| + \sum_{i=1}^{n} \frac{deg^2(v_i)}{2},$$

where $deg(v_i)$ is the number of edges incident with v_i.

We consider the edge ideal $\mathcal{I}$ of B_{2t} and we give an expression for the second Betti number b of $R/\mathcal{I}$ in terms of the number of the regions of B_{2t}.

Theorem 1.3 *Let B_{2t} be the bipartite planar graph, $r = 2t$ be the number of its regions and $\mathcal{I}$ be the edge ideal. If*

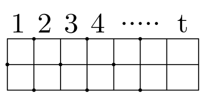
$$\ldots \to R^c(-4) \oplus R^b(-3) \to R^q(-2) \to R \to R/\mathcal{I} \to 0$$

is the minimal graded resolution of $R/\mathcal{I}$, then:

1) $q = \frac{5}{2}r + 2$;

2) $b = 6r - 2$.

Proof: 1) $q = |E(B_{2t})| = |\{\{v_i, v_{i+1}\} : \quad 1 \leq i \leq 3t+2, \quad i \neq t+1, t+2\}| + |\{\{v_i, v_{i+t+1}\} : \quad 1 \leq i \leq 2t+2\}| = (3t+2-2) + (2t+2) = 5t+2 = \frac{5}{2}r + 2$
2) By Theorem 1.2, $b = |E(L(B_{2t}))| - N_3$, where $N_3 = 0$ because the graph is bipartite. One has by Remark 1.5:
$|E(L(B_{2t}))| = -|E(B_{2t})| + \sum_{i=1}^{N} \frac{deg^2(v_i)}{2}$, where $N = 3t + 3$.
We observe that B_{2t} has $N = 3(t + 1)$ vertices representable in the plane on three horizontal lines and on each line there are $t+1$ vertices. Infact the representation in the plane of B_{2t} is a sequence of squares without chords disposed in 2 rows and t columns:



It follows that:
$\sum_{i=1}^{3t+3} \frac{deg^2(v_i)}{2} = 4(\frac{2^2}{2}) + 2t(\frac{3^2}{2}) + (t-1)(\frac{4^2}{2}) = 17t = \frac{17}{2}r$,
where $deg(v_1) = deg(v_{t+1}) = deg(v_{2t+3}) = deg(v_{3t+3}) = 2$,
$deg(v_i) = 3$ for $2 \leq i \leq t$, $i = t+2, 2t+2$ and $2t+4 \leq i \leq 3t+2$
$deg(v_i) = 4$ for $t+3 \leq i \leq 2t+1$.
Then:

$$b = |E(L(B_{2t}))| = -(\frac{5}{2}r + 2) + \frac{17}{2}r = 6r - 2.$$

Remark 1.6 *Let $G = B_{2t}$ and $\mathcal{I}$ be its edge ideal. The minimal graded resolution of $R/\mathcal{I}$ is said the minimal graded resolution of B_{2t} and the Betti numbers that appear in this resolution are the Betti numbers associated to B_{2t}.*

Example 1.3 *$G = B_6$, $V = \{v_1, \ldots, v_{12}\}$ and $\mathcal{I}$ its edge ideal. The minimal graded resolution of B_6 is:*

$$\ldots \to R^{45}(-4) \oplus R^{34}(-3) \to R^{17}(-2) \to R \to R/\mathcal{I} \to 0$$

By theorem 1.3 one has: $q = \frac{5}{2}r + 2 = 17$ and $b = 6r - 2 = 34$.

2 Bounds in the resolution of B_{2t}

We are interested to find bounds for the graded Betti numbers that appear in the minimal graded resolution of B_{2t}, in particular we give upper bounds for them in terms of the number of the regions of B_{2t}.
In [6] it is proved that the $i-th$ graded Betti numbers associated to a subgraph can not exceed the $i-th$ graded Betti numbers of the larger graph for all i.

Proposition 2.1 *([6], 4.1.1)*
Let G be a graph. If H is a subgraph of G on a subset of the vertices of G, with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, then

$$b_{i_j}(H) \leq b_{i_j}(G),$$

where $b_{i_j}(H)$ (resp.$b_{i_j}(G)$) are the graded Betti numbers of H (resp. G).

Proposition 2.2 *Let B_{2t} be the bipartite planar graph, $r = 2t$ be the number of its regions and $\mathcal{I}$ be the edge ideal. Let $b_{i_j}(B_{2t})$ be the graded Betti numbers in the minimal graded resolution of $R/\mathcal{I}$. Then:*
1) $b_{i_j}(B_{2t}) < \sum_{k+l=i+1,k,l\geq 1} \binom{\frac{3r+8}{4}}{k}\binom{\frac{3r+4}{4}}{l}$, *if t is even;*
2) $b_{i_j}(B_{2t}) < \sum_{k+l=i+1,k,l\geq 1} \binom{\frac{3r+6}{4}}{k}\binom{\frac{3r+6}{4}}{l}$, *if t is odd.*

Proof: B_{2t} is a bipartite planar graph on two disjoint vertex set V_1 and V_2, but it is not complete. Moreover it is a subgraph of the complete bipartite graph on the same vertex sets.
1) If t is even by Remark 1.3 we have $|V_1| = \frac{3t+4}{2}$ and $|V_2| = \frac{3t+2}{2}$. Then B_{2t} is a proper subgraph of the complete bipartite graph $K_{n,m}$, where $n = \frac{3t+4}{2}$ and $m = \frac{3t+2}{2}$, that is $V(B_{2t}) = V(K_{n,m})$ and $|E(B_{2t})| < |E(K_{n,m})|$. Then by Proposition 2.1 we have: $b_{i_j}(B_{2t}) < b_{i_j}(K_{n,m})$, where $b_{i_j}(K_{n,m})$ are the graded Betti numbers of $R/I(K_{n,m})$. By [6](Theorem 5.2.4) we have: $b_{i_j}(K_{n,m}) = \sum_{k+l=i+1,k,l\geq 1} \binom{n}{k}\binom{m}{l}$.
It follows:

$$b_{i_j}(B_{2t}) < \sum_{k+l=i+1,j,l\geq 1} \binom{\frac{3t+4}{2}}{k}\binom{\frac{3t+2}{2}}{l}, \quad where \ \ t = \frac{r}{2}.$$

2) If t is odd by Remark 1.3 we have $|V_1| = |V_2| = \frac{3t+3}{2}$. If follows that B_{2t} is a proper subgraph of the complete bipartite graph $K_{n,m}$, where $n = m = \frac{3t+3}{2}$, that is $V(B_{2t}) = V(K_{n,m})$ and $|E(B_{2t})| < |E(K_{n,m})|$. As before we obtain

$$b_{i_j}(B_{2t}) < \sum_{k+l=i+1,k,l\geq 1} \binom{\frac{3t+3}{2}}{k}\binom{\frac{3t+3}{2}}{l} \ \ where \ \ t = \frac{r}{2}.$$

The thesis follows.

In particular for the third Betti number in degree four studied in [4] we give the following result:

Corollary 2.1 *Let B_{2t} be the bipartite planar graph on $3t+3$ vertices, $r = 2t$ be the number of its regions and $\mathcal{I}$ be the edge ideal. Let*

$$\ldots \to \ldots \oplus R^d(-4) \to R^c(-4) \oplus R^b(-3) \to R^q(-2) \to R \to R/\mathcal{I} \to 0$$

be the minimal graded resolution of $R/\mathcal{I}$. Then:
1) $d < \frac{3}{64}r(\frac{3}{2}r+4)(\frac{3}{2}r+2)(\frac{7}{2}r+2)$, if t is even;
2) $d < \frac{3}{64}(\frac{r}{2}+1)^2(\frac{3}{2}r+1)(\frac{21}{2}r+10)$, if t is odd.

Proof: 1) If t is even one has:
$d = b_{3_4}(B_{2t}) < \sum_{k+l=4} \binom{\frac{3t+4}{2}}{k}\binom{\frac{3t+2}{2}}{l} = \binom{\frac{3t+4}{2}}{1}\binom{\frac{3t+2}{2}}{3} + \binom{\frac{3t+4}{2}}{2}\binom{\frac{3t+2}{2}}{2} + \binom{\frac{3t+4}{2}}{3}\binom{\frac{3t+2}{2}}{1} = \frac{3}{32}t(3t+4)(3t+2)(7t+2) = \frac{3}{64}r(\frac{3}{2}r+4)(\frac{3}{2}r+2)(\frac{7}{2}r+2)$.
2) If t is odd one has:
$d = b_{3_4}(B_{2t}) < \sum_{k+l=4} \binom{\frac{3t+3}{2}}{k}\binom{\frac{3t+3}{2}}{l} = \binom{\frac{3t+3}{2}}{1}\binom{\frac{3t+3}{2}}{3} + \binom{\frac{3t+3}{2}}{2}\binom{\frac{3t+3}{2}}{2} + \binom{\frac{3t+3}{2}}{3}\binom{\frac{3t+3}{2}}{1} = \frac{3}{64}(t+1)^2(3t+1)(21t+10) = \frac{3}{64}(\frac{r}{2}+1)^2(\frac{3}{2}r+1)(\frac{21}{2}r+10)$.

Now we compute the dimension of $R/\mathcal{I}$ and we find bounds for the projective dimension of $R/\mathcal{I}$ using the geometry of the planar graph B_{2t}.

Definition 2.1 *Let G be a graph with vertex set V. A subset $\mathcal{A}$ of V is said minimal vertex cover for G if each edge of G is incident with one vertex in $\mathcal{A}$ and there is no proper subset of $\mathcal{A}$ with this property.*

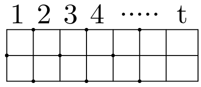
Definition 2.2 *The smallest number of vertices in any minimal vertex cover of G is said vertex covering number. We denote it $\alpha_0(G)$.*

Proposition 2.3 *Let B_{2t} be the bipartite planar graph with $r = 2t$ regions and $t \geq 1$. Then:*

$$\alpha_0(B_{2t}) = \begin{cases} \frac{3}{4}r + \frac{3}{2} & \text{if } t \text{ odd} \\ \frac{3}{4}r + 1 & \text{if } t \text{ even} \end{cases}$$

Proof: By definition of B_{2t} we have $V(B_{2t}) = \{v_1, \ldots, v_{3t+3}\}$ and edge set $E(B_{2t}) = \{\{v_i, v_{i+1}\} | 1 \leq i \leq 3t+2, \quad i \neq t+1, t+2\} \cup \{\{v_i, v_{i+t+1}\} | 1 \leq i \leq 2t+2\}$. Hence its representation in the plane is a sequence of squares without chords disposed in 2 rows and t columns:

1 2 3 4 ····· t



As see in the picture, $\alpha_0(B_2) = 3$ for $t = 1$ and in general $\alpha_0(B_{2t})$ is given by $\alpha_0(B_2)$ adjoining 1 vertex for each even column and 2 vertices for each odd column. Hence we compute:
1) If t is odd

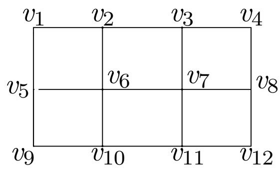
$$\alpha_0(B_{2t}) = \alpha_0(B_2) + 1(\frac{t-1}{2}) + 2(\frac{t-1}{2}) = \frac{3}{2}t + 3 = \frac{3}{4}r + \frac{3}{2},$$

where $\frac{t-1}{2}$ is the number of the even (odd) columns in the graph for $t > 1$.
2) If t is even

$$\alpha_0(B_{2t}) = \alpha_0(B_2) + 1(\frac{t}{2}) + 2(\frac{t}{2} - 1) = \frac{3}{2}t + 1 = \frac{3}{4}r + 1,$$

where $\frac{t}{2}$ is the number of the even columns and $\frac{t}{2} - 1$ is the number of the odd columns of the graph for $t > 1$.

Example 2.1 *$G = B_6$, with $V(B_6) = \{v_1, \ldots, v_{12}\}$ and*
$E(B_6) = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_5, v_6\}, \{v_6, v_7\}, \{v_7, v_8\}, \{v_9, v_{10}\}, \{v_{10}, v_{11}\}, \{v_{11}, v_{12}\}, \{v_1, v_5\}, \{v_2, v_6\}, \{v_3, v_7\}, \{v_4, v_8\}, \{v_5, v_9\}, \{v_6, v_{10}\}, \{v_7, v_{11}\}, \{v_8, v_{12}\}\}$



We have $\alpha_0(B_6) = 6$, in fact $\mathcal{A}(B_6) = \{v_2, v_4, v_5, v_7, v_{10}, v_{12}\}$.

Proposition 2.4 *Let B_{2t} be the bipartite planar graph with $r = 2t$ regions, $t \geq 1$ and $\mathcal{I}$ be the edge ideal. Then:*

$$dim(R/\mathcal{I}) = \begin{cases} \frac{3}{4}r + \frac{3}{2} & if\ t\ odd \\ \frac{3}{4}r + 2 & if\ t\ even \end{cases} .$$

Proof: Let $R = K[X_1, \ldots, X_n; Y_1, \ldots, Y_m]$ and $\mathcal{I} \subset R$ be the edge ideal of B_{2t} with $|V(B_{2t})| = n + m = 3t + 3$. By [7] (2.1.7) we have $dim(R/\mathcal{I}) = dim(R) - ht(\mathcal{I})$ and by [7] (6.1.18) $ht(\mathcal{I}) = \alpha_0(B_{2t})$. Hence $dim(R/\mathcal{I}) = (n + m) - \alpha_0(B_{2t}) = 3t + 3 - \alpha_0(B_{2t})$. Then by Proposition 2.3 it follows:
1) $dim(R/\mathcal{I}) = \frac{3}{2}r + 3 - (\frac{3}{4}r + \frac{3}{2}) = \frac{3}{4}r + \frac{3}{2}$, if t is odd,
2) $dim(R/\mathcal{I}) = \frac{3}{2}r + 3 - (\frac{3}{4}r + 1) = \frac{3}{4}r + 2$, if t is even.

Proposition 2.5 *Let B_{2t} be the bipartite planar graph with $r = 2t$ regions and $\mathcal{I}$ be the edge ideal. Then:*
1) $\frac{3}{4}r + \frac{3}{2} < pd_R(R/\mathcal{I}) < \frac{3}{2}r + 2$, if t is odd;
2) $\frac{3}{4}r + 1 < pd_R(R/\mathcal{I}) < \frac{3}{2}r + 2$, if t is even.

Proof: For the lower bounds by [7] (2.5.14) one has $pd_R(R/\mathcal{I}) > ht(\mathcal{I})$. Hence by [7] (6.1.18) it follows $pd_R(\mathcal{I}) > \alpha_0(B_{2t})$. Then the thesis follows by Proposition 2.3.
For the upper bounds we observe that B_{2t} is a proper subgraph of the bipartite complete graph $K_{n,m}$, with $V(B_{2t}) = V(K_{n,m})$, that is $n + m = 3t + 3$ and $|E(B_{2t})| < |E(K_{n,m})|$.
The projective dimension of a graph is affected by some simple transformations such as deleting some edges. So as a consequence of [6] (4.1.3) we have $pd_R(R/\mathcal{I}) < pd_R(R/I(K_{n,m})) = n + m - 1$ ([6], 4.2.9), with $n + m - 1 = 3t + 3 - 1 = \frac{3}{2}r + 2$. Hence: $pd_R(R/\mathcal{I}) < \frac{3}{2}r + 2$.

Example 2.2 *$G = B_6$ and $\mathcal{I}$ the edge ideal of B_6. Then: $dim(R/\mathcal{I}) = 6$ and $6 < pd_R(R/\mathcal{I}) < 11$.*

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SAFETY AND EFFICIENCY ANALYSIS OF A RAILWAY NETWORK

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Abstract. Today an optimal management of railway network constitutes a suitable and continuous requirement and it is necessary to assure to the users suitable standards of safety and comfort.
It is, therefore, necessary that the Companies that have the management of the infrastructures equip themselves of an adaptive and cognitive instrument, of dynamic type, of the network, so to have an effective technical support for the decisions, at the various levels of competence.
In the present paper, an opportune methodology of approach to the problem is described, where all is led back the construction of an informative network system, through an adapted mathematical instrument, to the definition of one particular matrix $\mathbf{X} \equiv \|x_{hk}\|$, with $h = 1, \ldots\ldots, n$; $k = 1, \ldots\ldots, m$, where n is the number of the system variables under study (homogenous arcs of the network, nodes, routes, etc.) and m is the number of the considered characteristic indicators.

1. Introduction

Improving the management of a complex system of infrastructures is one of the key elements of the programmatic aims of Agencies owners of the infrastructure, in order to achieve appropriate cost containment and, at the same time, to minimize the costs of maintenance ensuring to users suitable standards of safety and comfort.

Today, thanks to the advanced state of research in road and rail issues and to modern technologies, it is possible to proceed appropriately and in a timely way to analyze planned maintenance requirements of the network, analysing the data sets derived from the values of specific indicators, which appropriately held under control, allow to achieve the goal of the perfect functionality of infrastructures and, at the same time, to guaranty high levels of reliability for the entire network.

However, the great amount of parameters that are necessary to monitor for a continuous and as much as possible complete knowledge of true conditions of all the system components (nodes, links, etc.) makes the optimal management of the network often difficult and complex.

We must, in fact, show that this quantity of information, even if it were available, it is not always of easy and immediate interpretation, because the variables considered refer to various operational aspects which do not easily compare with each other.

This problem assumes strategic importance in the network context, where it is particularly difficult to identify and extract the dangerous elements which can represent the primary cause of the accident phenomena.

In the light of this, therefore, it is essential the study of an appropriate methodological approach that allows a complete information management in order to eliminate or minimize abnormal situations that may increase the level of localized or widespread risk.

2. A mathematical characterisation of the quality of the offer

The functionality of a network depends on a number of factors of different nature with a lot of interdependences that, on the whole, trace back to the analysis of the system: *"man-environment-infrastructure-vehicle"* [1][2].

In order to ensure in all operational conditions appropriate standards of safety and efficiency it is essential to monitor in real time the functionality of the network, in order to identify those critical situations which require immediate interventions of maintenance and/or management in order not to compromise the global level of safety.

The Agency managers of network must, consequently, be equipped with appropriate models of analysis, which through a more or less complex processing of the input data, allow to evaluate, in quantitative terms, the efficiency level of the system.

So, the functional quality of lines and nodes of the railway system, indicating with x_{hk} the generic variable among the m fully monitored, can be represented in the m-dimensional space of reference $\Re^m$ through a matrix

$$\mathbf{X} \equiv \left\| x_{hk} \right\|$$

with $h = 1, \ldots, n$; $k = 1, \ldots, m$, where n are the components of the considered system (homogeneous arch of the network, nodes, links, etc.) of the operating management optimization in terms of safety and operative efficiency.

In fact, if for the generic h-th component of the network we indicate with:

- $\vec{\mathbf{v}}_{\mathbf{q}}$ the vector of the performance and railway quality characteristics (geometry of the axis, security and comfort of march, etc.);
- $\vec{\mathbf{v}}_{\mathbf{t}}$ the representative vector of the efficiency state of the infrastructure (track, superstructure, operating average speed, etc.);
- $\vec{\mathbf{v}}_{\mathbf{o}}$ the vector characterising the functionality and the eventual deficiencies of the technological works and the equipment of line and station;
- $\vec{\mathbf{v}}_{\mathbf{n}}$ the vector characterising the operation of the fixed and mobile equipment (deficit of the wagons, reliability of the components of the equipment and signalling system for the line buffers, etc.);
- $\vec{\mathbf{v}}_{\mathbf{g}}$ the vector representative of the efficiency state of the infrastructure;
- $\vec{\mathbf{v}}_{\mathbf{e}}$ the vector of the existing relations between environmental conditions them (snow, ices, fog, fires, landslides, etc.) and the efficiency state of the infrastructure;
- $\vec{\mathbf{v}}_{\mathbf{s}}$ the characteristic vector of the interactions with the context of the

multimodal system to which it belongs,
we can totally refer to the vector $\vec{\mathbf{x}}$ *(union set of the considered classes)*:
we have:

$$\vec{\mathbf{x}} = \vec{\mathbf{v}}_{\mathbf{q}} \cup \vec{\mathbf{v}}_{\mathbf{t}} \cup \vec{\mathbf{v}}_{\mathbf{o}} \cup \vec{\mathbf{v}}_{\mathbf{g}} \cup \vec{\mathbf{v}}_{\mathbf{e}} \cup \vec{\mathbf{v}}_{\mathbf{s}}$$

Let us consider the general case in which the vector $\vec{\mathbf{x}}$ has components x_k, with $k = 1$,…, m, each of these vectors will be formed by elements x_{hk}, with $h = 1$,…, n , relative to the every observed variable k-th.

So we can define the matrix $\mathbf{X}_{nxm}$, constituted by column vectors of order n (equal to the number of the homogenous components of the network).

Defined the matrix representative of the functional state of the infrastructure, it is necessary to define an appropriate algorithm to achieve a synthesis of the several measured data, in order to have an effective analysis tool for the optimization of the maintenance and retraining works [3].

3. The proposed model

The operating approach illustrated in the following, by the employment of typical concepts of the multivariate techniques and logic fuzzy, has the aim to define a special hierarchical structure of dominance, easy interpretable with consolidated valuation methods like the Analytic Hierarchy Process - AHP (Saaty 1980) [4][5][6].

Among the various types of multivariate analysis, in particular, the *factorial analysis of the principal component* is used [7].

The factorial analysis is a suitable statistical method to reduce a complex system of correlations into one of smaller dimensions, so it is possible to achieve an opportune descriptive economy of the observed phenomena and processes; therefore, the objective of the *principal component analysis* (PCA) is to reduce a set of information into its principal components minimizing loss of knowledge.

PCA involves calculations of eigenvalues and their corresponding eigenvectors of the covariance (or correlation) matrix to derive the new variables in a decreasing order of importance in explaining variation of the original variables. Usually, if correlations among the original variables are large enough, the first few components will account for most of the variation in the original data. If that is the case, then they can be used to represent the data with little loss of information, thus providing a suitable way in reducing the dimensionality of the data.

The principal components (PCs) are calculated as linear combinations of the original variables. Each component is characterized characterised by an eigenvalue, which represents the rate of variance explained by the component; the first extracted components have the more elevated eigenvalues and i.e. they better synthesise the contained information in the original matrix.

The interpretation of the PCs takes place in their correlations with the original variables: each variable presents, in fact, a correlation with a particular principal component, but some PCs will have higher correlations with a specific component.

Higher the correlation is the more variable is associated with that PC.

Starting from the built matrix **X**, we determine the correlation matrix **C**, given by the product of the transposed matrix **X** for itself:

$$\mathbf{C} = {}^{t}\mathbf{X} \cdot \mathbf{X}$$

The calculation of the equation characteristic roots

$$\left|\mathbf{C} - \lambda \mathbf{I}\right| = \mathbf{0}$$

where **I** is the identity matrix of m dimension m and **0** is a column vector of zeros, will allow to calculate the eigenvalues λ_h ($h = 1 \ldots m$, con $\lambda_h \geq 0$).
And after ordering the eigenvalues in a decreasing order

$$\lambda_1 \geq \lambda_2 \geq .... \geq \lambda_m$$

We can calculate the corresponding eigenvectors from the equation

$$\mathbf{C} - \lambda_h \mathbf{I} \ \cdot \mathbf{g_h} = 0$$

The vectors $\mathbf{g_1}$, $\mathbf{g_2}$, …, $\mathbf{g_m}$ determine the factorial axis of the correlation matrix and their PCs are represented by m uncorrelated linear functions [7][8].

Each of these variables is a linear combination of the original variables and can be considered as an "*artificial variables*", that is:

$$f_i = \sum_{k=1}^{m} g_{ki} \cdot x_k$$

where f_i ($i = 1, \ldots, m$) is the i-th *principal components*.

The use of the correlation matrix instead of the covariance matrix in the PCA was to assign equal weights for all of the indicators in the analysis in forming the principal components.

Now, the problem is to decide which and how principal components to use in order to obtain an efficient synthesis of data.

One of the used criterions is to consider all the components with eignvalue $\lambda_h \geq 1$ (Guttman, 1954; Kaiser, 1960) [7][9].

In general, the number of factors with eignvalue greater of 1 changes between 1/6 and 1/3 of the considered variables; so we have a consistent synthesis reducing the number of variables form m to s (with $s < m$) without any considerable loss of information.

Analysing the correlation matrix between the determinate principal components and the original variables it is possible to estimate the factorial weights, finding for each principal components the associated indicators.

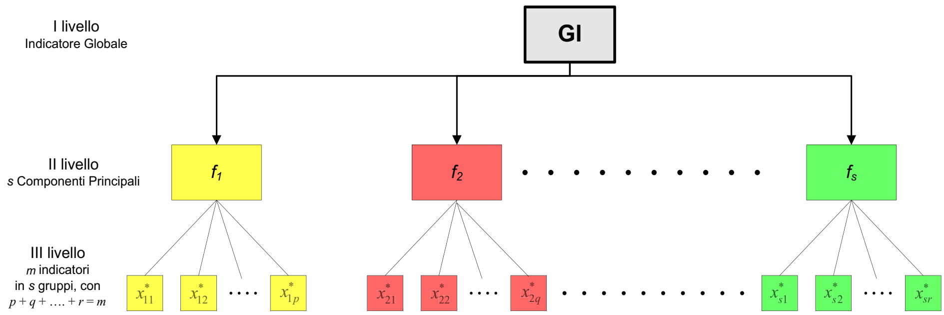


Figure 1 – Three levels AHP hierarchy

Then, for each network components it is possible to build a hierarchical structure of three levels (figure 1): the highest level has the wanted global indicator that characterizes the functionality of the analysed network components; the second one has the primes *s* PCs and the third has the *m* original variables, each of which is associated with the PC where it has the highest factorial weight.

After the hierarchical structure have been built it is necessary to normalize and, if necessary, scale-reverse all the indicators, so to have them all on the same 0-1 scale, with 0 and 1 representing the ideal and undesirable reference points, respectively [9].

Then, all the elements x_{hk} of $\vec{\mathbf{x}}$ are transformed in x^*_{hk} elements, with $0 \leq x^*_{hk} \leq 1$.

In the AHP method, the step after constructing the hierarchy is to carry out pairwise comparison judgments at different levels of the hierarchy to reveal the criteria's relative weights [10][11].

In our models, a weight is assigned for the *s* PCs at the second level using the amount of variation explained.

Weight at the third for indicators associated with the same PC are equally assigned (i.e., for all the indicators associated to the component f_1 the weight is equal to $1/p$, for those one associated to the component f_2 it is equal to $1/q$ and so on until the last component f_s).

The so esteemed weights are *local weight* because they measure the importance of the elements not in complex terms, but only in relation to the corresponding higher element of hierarchy.

To evaluate the importance of each element in relation to highest level we need to multiply the local weights of each item to the corresponding upper.

Proceeding from top to bottom, the local weights of all the elements of hierarchy are transformed gradually into global weights.

The global weight W of the principal components is equal to the local one (in function of the amount of explained variance) because the only element at the top level has obviously a weigh equal to one.

Instead, at the third level we have:
for the p indicators associated to the component f_1

$w_{11} = w_{12} = \cdots = w_{1p} = W_1 \cdot \left(\frac{1}{p}\right)$, with W_1 the weight of the PC f_1;

for the q indicators associated to the component f_2

$w_{21} = w_{22} = \cdots = w_{2q} = W_2 \cdot \left(\frac{1}{q}\right)$, with W_2 the weight of the PC f_2;

…….

for the r indicators associated to the component f_s

$w_{s1} = w_{s2} = \cdots = w_{sr} = W_s \cdot \left(\frac{1}{r}\right)$, with W_s the weight of the PC f_s.

The used methodology represents only one of the possible solutions for the computation of weights that, in general, can be estimated using other procedures well known in the literature.

1.1. Model analysis

The n hierarchies allow the valuation of the functional state of the considered observation units (arches, nodes, etc.) by computing for each indicator the fuzzy distance from the ideal state.

We use the theory of fuzzy logic because many of the considered indicators are not suited to quantify/valute by classical procedures, in fact in some cases they can be variable inside the observation unit, in others, they can be only estimated with an error margin.

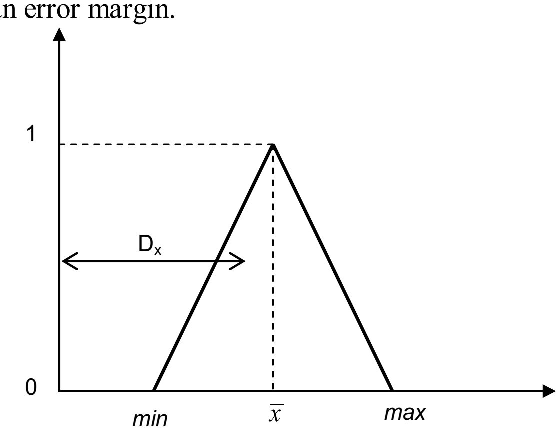


Figure 2 – Triangle fuzzy number

We intend to construct a *triangle fuzzy number* for an indicator by using its value and its possible range (i.e., minimum and maximum values) inside the observed unit.

The fuzzy distance between the ideal state (representing by the triangular fuzzy number $(a_1, a_2, a_3)_T$, with $a_1 = a_2 = a_3 = 0$, and the fuzzy value of every indicators $(b_1, b_2, b_3)_T$ is calculated by the following expression [12]:

$$D_x^2 = b_2^2 + \frac{1}{3} b_2 \cdot (b_3 + b_1) + \frac{1}{18}\left[(b_3 - b_2)^2 + (b_2 - b_1)^2 \right] - \frac{1}{18}\left[(b_2 - b_1)(b_3 - b_2) \right]$$

At last, the scores (or distances) computed at the lowest level are aggregated at other higher levels as weight sum of the scores:

$$D_{\text{level } j}^{\text{variable } f_s} = \sum_{k=1}^{r} w_{sk} \cdot D_{\text{level } j+1}^{\text{variable } x_{sk}^*}$$

where $D_{\text{livel } j}^{\text{variable } f_s}$ is the distance of variable f_s at level j, w_{sk} is the local weight of the variable x_{sk}^* at level $j+1$ and r is the number of the indicators at level j+1 associated with the variable f_s at level j.

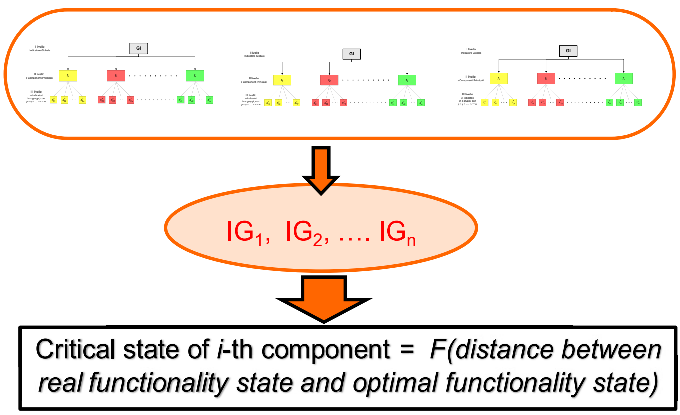
As example, for the first principal component we have:

$$D_{\text{level } 2}^{\text{variable } f_1} = \sum_{k=1}^{p} w_{1k} \cdot D_{\text{level } 3}^{\text{variable } x_{1k}^*}$$

The distances at an upper level are evaluated as weighted sum of distances at the lower level.

The ultimate score represents the distance of the i-th component from an ideal component that has the ideal states for all the indicators.

So the ultimate scores are used to derive a relative ranking for all the network components, which can be used to identify the most critical situation.



4. Conclusions

The management of a rail network needs a comprehensive and continuous monitoring of the offer quality, in order to optimize, in a dynamic way, the complex system “man-environmental-infrastructure-vehicle”.

Therefore, we must have an appropriate knowing tool of the real operating conditions and of the related need of maintenance interventions for the single components of the rail network, to ensure to users adequate standards of quality and safety.

In this memory, identified a methodology approach to the problem, for effective computerized management of information we elaborate a particular model for the study of the network quality that is based on vector algebra and logic fuzzy.

Represented the problem in $\Re_m$ space of measured indicators it is possible to calculate, by a particular algorithm, a special parameter characterizing the operating standards of network component, making easy the identification of critical situations that require particular attention.

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Dissipative processes in extrinsic semiconductors

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Abstract

In a previous paper, in a geometrized framework for the description of simple materials with internal variables, the specific example of extrinsic semiconductor crystals was considered and the relevant structure of the entropy 1-form was derived, taking into account a nonconventional model based on the extended irreversible thermodynamics. The aim of this paper is to contribute to describe the behaviour of extrinsic semiconductors. In particular, Clausius-Duhem inequality for these media is exploited, and, using Maugin's technique, the laws of state, the extra entropy flux and the residual dissipation inequality are worked out. Finally, following Maugin, the heat equation in the first and the second form is derived.

Introduction

In a previous paper [1] a geometrization technique for thermodynamics of simple materials [2]-[5] was applied to the specific example of extrinsic semiconductors (described in [6]-[8] in the framework of extended irreversible thermodynamics with internal variables), and a geometrical model for these media was derived. The dynamical system describing their behaviour, the morphism defined on the fibre bundle of the process, the transformation induced by the process and the entropy function and the entropy 1-form were derived. In this paper, taking in account the nonconventional model for extrinsic semiconductors developed in [6]-[8], we exploit the Clausius-Duhem inequality for these media and using Maugin's technique [9] (see also the Colemann-Noll procedure [10]), we work out the laws of state, the extra entropy flux, the residual dissipation inequality. Finally, following Maugin, the heat equation in the first and the second form is obtained. The models for semiconductors are applied in the technology of integrated circuits, in the field of electronic microscopy, in nanotechnology, in computer science and in many sectors of applied science. Semiconductor crystals, as germanium and silicon, are tetravalent elements [11]. In Fig.1_a we have the representation of a germanium crystal that has a behaviour of an insulator at a temperature of $0°K$. But, at room temperature (see Fig.1_b), because electrons of the crystal can gain enough thermal energy to leap the band gap from the valence band to the conduction band and to be available for the electric current conduction, it has a behaviour of a conductor. Then, to modify the electrical conductivity of an intrinsic semiconductor, impurity atoms adding one electron or one hole are introduced inside the crystal, by means of different techniques of "doping". Using pentavalent impurities, as antimony, an n-type extrinsic semiconductor is obtained, having more free electrons that may flow (see Fig.2_a). By trivalent impurities, as indium, an p-type extrinsic semiconductor crystal is achieved, having more holes that may flow freely (see Fig.2_b).

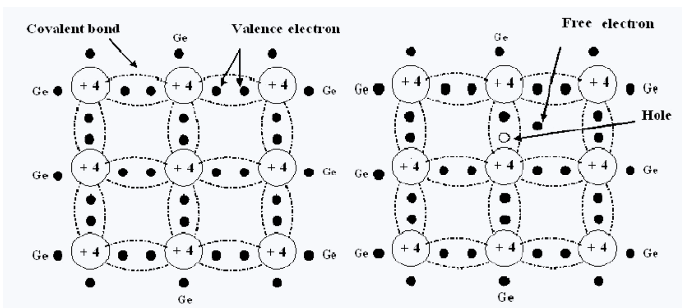

Figure 1: A symbolic representation in 2D of a germanium crystal structure: (a) at $0°K$ and (b) at $300°K$ with a broken covalent bond

1 A non conventional model for extrinsic semiconductors

In [6]-[8], in the framework of the extended rational thermodynamics with internal variables, using the standard Cartesian tensor notation in a rectan-

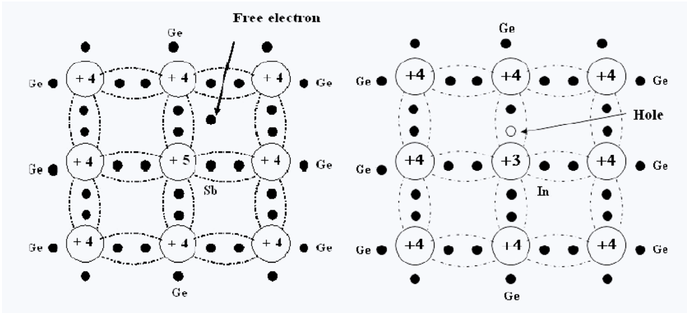

Figure 2: (a) A crystal structure with an atom of Germanium replaced with an atom of a pentavalent impurity (Antimony); (b) A crystal structure with an atom of Germanium replaced with an atom of trivalent impurity (Indium)

gular coordinate system, a model was developed for extrinsic semiconductors, in which the following fields interact with each other: *the elastic field* described by the total stress tensor T_{ij} and the small-strain tensor ε_{ij}; *the thermal field* described by the temperature θ, its gradient and the heat flux q_i; *the electromagnetic field* described by the electromotive intensity $\mathcal{E}_i$ and the magnetic induction B_i per unit volume; *the charge carrier fields* described by the densities of electrons n and holes p and their fluxes j_i^n and j_i^p. The independent variables are represented by the set

$$C = \{\varepsilon_{ij}, \mathcal{E}_i, B_i, n, p, \theta, j_i^n, j_i^p, q_i, n_{,i}, p_{,i}, \theta_{,i}\}. \tag{1}$$

This specific choice shows that the relaxation properties of the thermal field, the and charge carrier fields are taken into account. However, we ignore the corresponding effect for the mechanical properties so that T_{ij} is not in the set (1). All the processes, occurring in the considered body, are governed by the following laws:

Maxwell's equations :

$$\varepsilon_{ijk} E_{k,j} + \frac{\partial B_i}{\partial t} = 0, \quad D_{i,i} - \rho Z = 0, \tag{2}$$

$$\varepsilon_{ijk} H_{k,j} - j_i - \rho Z v_i - \frac{\partial D_i}{\partial t} = 0, \quad B_{i,i} = 0, \tag{3}$$

where **E**, **B**, **D** and **H** denote the electric field, the magnetic induction, the electric displacement and the magnetic field per unit volume, respectively. Moreover,

$$H_i = \frac{1}{\mu_0} B_i, \qquad E_i = \frac{1}{\varepsilon_0} \left(D_i - P_i \right), \tag{4}$$

v_i is the velocity of the body point, ε_0, μ_0 denote the permittivity and permeability of vacuum, $\mathbf{P}$ is the electric polarization per unit volume and the total charge Z and current $\mathbf{j}$ are as follows

$$Z = n + \bar{n} - n_0 + p + \bar{p} - p_0, \quad j_i = j_i^n + j_i^p,$$

with $n < 0$, $n_0 < 0$, $p > 0$, $p_0 > 0$, $\bar{n} < 0$ and $\bar{p} > 0$ denoting mass densities of nonequilibrium and equilibrium electrons and holes, and the charges of ionized impurities, respectively. Moreover, we assume that the magnetic properties of the semiconductor are discarded so that the magnetization of the body $M_i = 0$;

the *evolution equations of charge carriers* read:

$$\rho\dot{n} + j^n_{i,i} - g^n = 0, \quad \rho\dot{p} + j^p_{i,i} - g^p = 0,$$

where the superimposed dot denotes the material derivative and

the equations for the ionized impurities are as follows:

$$\rho\dot{\bar{n}} = \bar{g}^n, \quad \rho\dot{\bar{p}} = \bar{g}^p,$$

where g^n and g^p describe the recombination of electrons and holes and together with the ionization of impurities $\bar{g}^n$ and $\bar{g}^p$ satisfy the equation $g^n + g^p + \bar{g}^n + \bar{g}^p = 0$. Following [6]-[8], we assume $\dot{\bar{n}} = \dot{\bar{p}} = 0$ and $\bar{g}^n = \bar{g}^p = 0$. Furthermore, we have

the *evolution equations* concerning *electron*, *hole* and *heat fluxes*:

$$\overset{*}{j}{}^n_i - J^n_i(C) = 0, \qquad \overset{*}{j}{}^p_i - J^p_i(C) = 0, \qquad \overset{*}{q}_i = Q_i(C),$$

where

$$\overset{*}{j}{}^n_i = \dot{j}^n_i - \Omega_{ik} j^n_k, \qquad \overset{*}{j}{}^p_i = \dot{j}^p_i - \Omega_{ik} j^p_k, \qquad \overset{*}{q}_i = \dot{q}_i - \Omega_{ij} q_j,$$

with $\Omega_{ij} = \frac{1}{2}(v_{i,j} - v_{j,i})$ the antisymmetric part of the velocity gradient, and $\mathbf{J}^n$, $\mathbf{J}^p$ and $\mathbf{Q}$ being the electron, hole and heat flux sources. In the above equations a superimposed asterisk indicates the Zaremba-Jaumann time derivative (see [12]-[14] for the form of these equations);

the continuity equation:

$$\dot{\rho} + \rho v_{i,i} = 0,$$

where ρ denotes the mass density and the mass charge carriers have been neglected compared to ρ;

the momentum balance:

$$\rho\dot{v}_i - T_{ji,j} - \rho Z\mathcal{E}_i - \varepsilon_{ijk}\left(j_j^n + j_j^p + \overset{\triangle}{P}_j\right)B_k - P_j\mathcal{E}_{i,j} - f_i = 0,$$

where

$$\overset{\triangle}{P}_i = \dot{P}_i + P_i v_{k,k} - P_k v_{i,k}, \quad \mathcal{E}_i = E_i + \varepsilon_{ijk} v_j B_k, \tag{5}$$

T_{ij} denotes the total stress tensor, f_i is the body force (neglected in the following) and the field $\boldsymbol{\mathcal{E}}$ is the same field as $\mathbf{E}$, in the Galilean approximation, but referred to an element of the matter at time t, i.e. the so called comoving frame $\mathcal{K}_c$;

the moment of momentum balance:

$$\varepsilon_{ijk}T_{jk} + c_i = 0,$$

where c_i is a couple for unit volume;

the internal energy balance

$$\rho\dot{e} - T_{ji}v_{i,j} - \left(j_j^n + j_j^p\right)\mathcal{E}_j - \rho\mathcal{E}_i\dot{\mathcal{P}}_i + q_{i,i} - \rho r = 0. \tag{6}$$

Here e is the internal energy density, r is the heat source distribution, neglected in the following,

$$P_i = \rho\mathcal{P}_i, \qquad v_{i,j} = L_{ij} = L_{(ij)} + L_{[ij]}, \tag{7}$$

where $L_{(ij)}$ and $L_{[ij]}$ are respectively the symmetric and antisymmetric part of the velocity gradient. Introducing the deformation gradient $\mathbf{F}$, also we have

$$\mathbf{L} = \nabla\mathbf{v} = \dot{\mathbf{F}}\mathbf{F}^{-1}. \tag{8}$$

All the admissible solutions of the proposed evolution equations should be restricted by the following *entropy inequality*:

$$\rho\dot{S} + J_{S_{k,k}} - \frac{\rho r}{\theta} \geq 0,$$

where S denotes the entropy per unit mass and $\mathbf{J}_S$ is the entropy flux associated with the fields of the set C given by

$$\mathbf{J}_S = \frac{1}{\theta}\mathbf{q} + \mathbf{k}, \tag{9}$$

with $\mathbf{k}$ an additional term called *extra entropy flux density.* In [6]-[8] the following constitutive functions $\mathbf{Z} = \tilde{\mathbf{Z}}(C)$ with

$$\mathbf{Z} = \left\{T_{ij}, P_i, c_i, e, g^n, g^p, J_i^n, J_i^p, Q_i, S, \Phi_i, \mu^n, \mu^p\right\}, \tag{10}$$

were obtained for extrinsic semiconductors in different cases by analyzing the entropy inequality and some constitutive relations were derived expanding the free energy in Taylor series with respect to a particular natural state.

2 Clausius-Duhem inequality analysis and heat equation

In this section, taking into account the results obtained in [6]-[8], we exploit the Clausius-Duhem inequality for extrinsic semiconductors media. Following Maugin [9] (see also Colemann-Noll procedure [10]), we consider the entropy inequality

$$\rho \dot{S} + J^s_{i,i} \geq 0, \tag{11}$$

which takes the following form, being $\theta > 0$

$$\rho\theta\dot{S} + \left(\theta J^s_i\right)_{,i} - J^S_i \theta_{,i} \geq 0, \tag{12}$$

with J^s given by equ.(9). Then, we analyze the dissipation inequality obtaining the constitutive equations and the first and the second form of heat equation. Thus, we introduce the free energy per unit volume

$$\Psi = e - \theta S - \frac{1}{\rho}\mathcal{E}_i P_i, \tag{13}$$

where Ψ is the free energy, e is the internal energy, P_i denotes the polarization and $\mathcal{E}_i$ is the electromotive intensity. By derivation with respect to time of the free energy Ψ we obtain the following result:

$$\rho\theta\dot{S} = \rho\dot{e} - \rho S\dot{\theta} - \rho\dot{\Psi} + \frac{1}{\rho}\dot{\rho}\mathcal{E}_i P_i - \dot{\mathcal{E}}_i P_i - \mathcal{E}_i \dot{P}_i. \tag{14}$$

Substituting equ.(14) in the entropy inequality we have

$$\rho\dot{e} - \rho S\dot{\theta} - \rho\dot{\Psi} + \frac{1}{\rho}\dot{\rho}\mathcal{E}_i P_i - \dot{\mathcal{E}}_i P_i - \mathcal{E}_i \dot{P}_i + \left(\theta J^s_i\right)_{,i} - J^s_i \theta_{,i} \geq 0 \tag{15}$$

and using the internal energy balance equation (6) we obtain

$$-\rho(\dot{\Psi} + \dot{\theta}S) + T_{ji}v_{i,j} + (j^n_i + j^p_i)\mathcal{E}_i - q_{i,i} + (\theta J^s_i)_{,i} - \theta_{,i}J^s_i - \dot{\mathcal{E}}_i P_i \geq 0. \tag{16}$$

Finally, taking into account equations $(7)_1$ and (9), we derive

$$\rho\dot{\mathcal{P}}_i = -\frac{1}{\rho}\dot{\rho}P_i + \dot{P}_i, \tag{17}$$

$$\left(\theta J^s_i\right)_{,i} = q_{i,i} + \left(\theta k_i\right)_{,i}, \tag{18}$$

$$(\theta k_i)_{,i} = \rho\left(\frac{\theta}{\rho}k_i\right)_{,i} + \left(\frac{\theta}{\rho}k_i\right)\rho_{,i}. \tag{19}$$

Moreover, using the following relations

$$\rho\left(\frac{\partial\Psi}{\partial n_{,i}}\right)\dot{n}_{,i}=\rho\left(\frac{\partial\Psi}{\partial n_{,i}}\dot{n}\right)_{,i}-\rho\left(\frac{\partial\Psi}{\partial n_{,i}}\right)_{,i}\dot{n}, \tag{20}$$

$$\rho\left(\frac{\partial\Psi}{\partial p_{,i}}\right)\dot{p}_{,i}=\rho\left(\frac{\partial\Psi}{\partial p_{,i}}\dot{p}\right)_{,i}-\rho\left(\frac{\partial\Psi}{\partial p_{,i}}\right)_{,i}\dot{p}, \tag{21}$$

$$L_{ij}=v_{i,j}=\dot{F_{ik}}\left(F_{kj}\right)^{-1},$$

and the material derivative of the free energy Ψ (constitutive function of the independent variables $\Psi=\Psi\left(F_{ij},\mathcal{E}_i,B_i,n,p,\theta,j_i^n,j_i^p,q_i,n_{,i},p_{,i},\theta_{,i}\right)$), from equ.(15) we obtain the following Clausius Duhem inequality:

$$\begin{gathered}\left(T_{ji}F_{kj}^{-1}-\rho\frac{\partial\Psi}{\partial F_{ik}}\right)\dot{F}_{ik}-\left(\rho\frac{\partial\Psi}{\partial\mathcal{E}_i}+P_i\right)\dot{\mathcal{E}}_i-\rho\frac{\partial\Psi}{\partial B_i}\dot{B}_i+\rho\mathcal{A}^n\dot{n}+\\ +\rho\mathcal{A}^p\dot{p}-\rho\left(\frac{\partial\Psi}{\partial\theta}+S\right)\dot{\theta}-\rho\frac{\partial\Psi}{\partial j_i^n}\dot{j}_i^n-\rho\frac{\partial\Psi}{\partial j_i^p}\dot{j}_i^p+\\ -\rho\frac{\partial\Psi}{\partial q_i}\dot{q}_i-\rho\frac{\partial\Psi}{\partial\theta_{,i}}\dot{\theta}_{,i}-\rho\left[\frac{\partial\Psi}{\partial n_{,i}}\dot{n}+\frac{\partial\Psi}{\partial p_{,i}}\dot{p}-\frac{\theta}{\rho}k_i\right]_{,i}+\\ +(j_i^n+j_i^p)\mathcal{E}_i+\left(\frac{\theta}{\rho}k_i\right)\rho_{,i}-J_i^s\theta_{,i}\geq 0.\end{gathered} \tag{22}$$

In equ.(22) we have introduced the following notations:

$$\mathcal{A}^n=\left[-\frac{\partial\Psi}{\partial n}+\left(\frac{\partial\Psi}{\partial n_{,i}}\right)_{,i}\right]=-\frac{\delta\psi}{\delta n},\qquad \mathcal{A}^p=\left[-\frac{\partial\Psi}{\partial p}+\left(\frac{\partial\Psi}{\partial p_{,i}}\right)_{,i}\right]=-\frac{\delta\psi}{\delta p},$$

where $\frac{\delta\psi}{\delta n}$ and $\frac{\delta\psi}{\delta p}$ are the "space" Euler-Lagrange derivative respect to n and p respectively (see [9]). As $T_{ji}F_{kj}^{-1}$, P_i, $\frac{\partial\Psi}{\partial B_i}$ and S are assumed not to depend on $\dot{F}_{ik}$, $\dot{\mathcal{E}}_i$, $\dot{B}_i$, $\dot{\theta}$ and $\dot{\theta}_{,i}$, while the remaining coefficients may in general depend on their respective factors, from inequality (22) we obtain the following laws of state

$$\rho\frac{\partial\Psi}{\partial F_{ik}}=T_{ji}F_{kj}^{-1},\qquad \frac{\partial\Psi}{\partial B_i}=0,\qquad \rho\frac{\partial\Psi}{\partial\theta_{,i}}=0, \tag{23}$$

$$\rho\frac{\partial\Psi}{\partial\mathcal{E}_i}=P_i,\qquad \frac{\partial\Psi}{\partial\theta}=-S. \tag{24}$$

At this point, assuming

$$\left(\frac{\partial\Psi}{\partial n_{,i}}\dot{n}+\frac{\partial\Psi}{\partial p_{,i}}\dot{p}-\frac{\theta}{\rho}k_i\right)_{,i}=0, \tag{25}$$

"it is astute to select"

$$k_i = \left(\frac{\partial \Psi}{\partial n_{,i}} \dot{n} + \frac{\partial \Psi}{\partial p_{,i}} \dot{p} \right) \frac{\rho}{\theta}, \tag{26}$$

so that Clausius-Duhem inequality reduces itself to the following *residual dissipation inequality*

$$\rho \mathcal{A}^n \dot{n} + \rho \mathcal{A}^p \dot{p} - \rho \frac{\partial \Psi}{\partial j_i^n} \dot{j}_i^n - \rho \frac{\partial \Psi}{\partial j_i^p} \dot{j}_i^p +$$

$$-\rho \frac{\partial \Psi}{\partial q_i} \dot{q}_i + (j_i^n + j_i^p)\, \mathcal{E}_i + \left(\frac{\theta}{\rho} k_i \right) \rho_{,i} - J_i^S \theta_{,i} \geq 0. \tag{27}$$

Very often this is split in two parts ("resulting thus in stronger conditions")

$$\Phi_{intr} = \rho \mathcal{A}^n \dot{n} + \rho \mathcal{A}^p \dot{p} - \Pi_i^n \dot{j}_i^n - \Pi_i^p \dot{j}_i^p - \Pi_i^Q \dot{q}_i + (j_i^n + j_i^p)\, \mathcal{E}_i + \left(\frac{\theta}{\rho} k_i \right) \rho_{,i} \geq 0, \tag{28}$$

with

$$\boldsymbol{\Pi}^n = \rho \frac{\partial \Psi}{\partial \mathbf{j}^n}, \qquad \boldsymbol{\Pi}^p = \rho \frac{\partial \Psi}{\partial \mathbf{j}^p}, \qquad \boldsymbol{\Pi}^Q = \rho \frac{\partial \Psi}{\partial \mathbf{q}}, \tag{29}$$

affinities, and

$$\Phi_{th} = -\mathbf{J}_s \cdot \nabla \theta \geq 0, \tag{30}$$

where, in some sense, we recognize the different qualitative nature of the two classes of dissipative processes. Φ_{intr} and Φ_{th} are the intrinsic and thermal dissipations, respectively [9]. Thus, we have derived the thermodynamical state laws, the entropy flux and the inequality governing dissipative processes having the standard bilinear form in terms of fluxes and associated forces

$$\sum_{\beta} X_\beta Y_\beta \geq 0,$$

as usual in classical thermodynamics of irreversible processes. Now, in order to obtain the heat equation, we observe that it is none other than a form of energy balance equation (6). Indeed, on using the free energy expression $\Psi = e - \theta S$, its time derivative and the laws of state in the energy balance equation or, equivalently, "just comparing entropy inequality (6) and the residual inequality" (27), we deduce *the first general form of heat equation*

$$\rho \theta \dot{S} + (\theta J_i^s)_{,i} = \Phi_{intr}, \tag{31}$$

where the intrinsic dissipation acts like a body source of heat.
Now, taking into account that the entropy S is a constitutive function

$S = S(F_{ij}, \mathcal{E}_i, B_i, n, p, \theta, j_i^n, j_i^p, q_i, n_{,i}, p_{,i}, \theta_{,i})$, and the state laws $(23)_2$, $(23)_3$ and $(24)_2$ we have

$$\dot{S} = -\left[\frac{\partial^2\psi}{\partial\mathbf{F}\partial\theta}\cdot\dot{\mathbf{F}} + \frac{\partial^2\psi}{\partial\boldsymbol{\mathcal{E}}\partial\theta}\cdot\dot{\boldsymbol{\mathcal{E}}} + \frac{\partial^2\psi}{\partial n\partial\theta}\dot{n} + \frac{\partial^2\psi}{\partial p\partial\theta}\dot{p} + \frac{\partial^2\psi}{\partial\theta^2}\dot{\theta} + \frac{\partial^2\psi}{\partial\mathbf{j}^n\partial\theta}\cdot\dot{\mathbf{j}^n}\right.$$
$$\left. + \frac{\partial^2\psi}{\partial\mathbf{j}^p\partial\theta}\cdot\dot{\mathbf{j}^p} + \frac{\partial^2\psi}{\partial\mathbf{q}\partial\theta}\cdot\dot{\mathbf{q}} + \frac{\partial^2\psi}{\partial\nabla n\partial\theta}\cdot\dot{\nabla n} + \frac{\partial^2\psi}{\partial\nabla p\partial\theta}\cdot\dot{\nabla p}\right]. \tag{32}$$

Finally, setting

$$\mathcal{C} = -\rho\theta\frac{\partial^2\psi}{\partial\theta^2}, \qquad \boldsymbol{\tau} = \rho\frac{\partial^2\psi}{\partial\mathbf{F}\partial\theta}, \qquad \mathbf{g} = \rho\theta\frac{\partial^2\psi}{\partial\boldsymbol{\mathcal{E}}\partial\theta}, \tag{33}$$

$$h^n = \rho\theta\frac{\partial^2\psi}{\partial n\partial\theta}, \qquad h^p = \rho\theta\frac{\partial^2\psi}{\partial p\partial\theta}, \qquad \mathbf{n} = \rho\theta\frac{\partial^2\psi}{\partial\mathbf{j}^n\partial\theta}, \tag{34}$$

$$\mathbf{p} = \rho\theta\frac{\partial^2\psi}{\partial\mathbf{j}^p\partial\theta}, \quad \mathbf{z} = \rho\theta\frac{\partial^2\psi}{\partial\mathbf{q}\partial\theta}\cdot\dot{\mathbf{q}}, \quad \mathbf{r} = \rho\theta\frac{\partial^2\psi}{\partial\nabla n\partial\theta}, \quad \boldsymbol{\eta} = \rho\theta\frac{\partial^2\psi}{\partial\nabla p\partial\theta}, \tag{35}$$

using the definitions of affinities $\mathbf{\Pi}^n$, $\mathbf{\Pi}^p$ and $\mathbf{\Pi}^Q$ (see equ.(29)) and substituting equ.(32) in (31), we obtain "the second form of the heat equation" in the following compact form:

$$C\dot{\theta} + \nabla\cdot(\theta\mathbf{J_S}) = \Phi_{te} + \Phi_{tel} + \Phi_{tc} + \Phi_q + \Phi_{ch} + \Phi_{elc} + \Phi_\rho. \tag{36}$$

In (36) $\Phi_{te}$is the dissipation given by the interaction between thermal and elastic phenomena, Φ_{tel} represents the dissipation due to the interaction between thermal and electric phenomena, Φ_{tc} gives the dissipation given by the interaction between thermal phenomena and phenomena connected with charge carriers and fluxes of charge carriers, Φ_q is the dissipation connected with the thermal phenomena, $\Phi_c$is the dissipation due to the charge carriers phenomena and phenomena connected with fluxes of charge carriers, Φ_{elc} represents the dissipation due to the interaction between electric and charge carriers phenomena, Φ_ρ is the dissipation due to the phenomena connected with the mass field and the extra entropy flux and they are given by the following expressions

$$\Phi_{te} = \theta\left(\boldsymbol{\tau}\cdot\dot{\mathbf{F}}\right), \quad \Phi_{tel} = \mathbf{g}\cdot\dot{\boldsymbol{\mathcal{E}}}, \quad \Phi_q = \left(\mathbf{z} - \mathbf{\Pi}^Q\right)\cdot\dot{\mathbf{q}}, \quad \Phi_{elc} = \mathbf{j}\cdot\boldsymbol{\mathcal{E}}, \tag{37}$$

$$\Phi_{tc} = h^n\dot{n} + h^p\dot{p} + \mathbf{n}\cdot\dot{\mathbf{j}^n} + \mathbf{p}\cdot\dot{\mathbf{j}^p} + \mathbf{r}\cdot\dot{\nabla n} + \boldsymbol{\eta}\cdot\dot{\nabla p}, \tag{38}$$

$$\Phi_{ch} = \rho\mathcal{A}^n\dot{n} + \rho\mathcal{A}^p\dot{p} - \mathbf{\Pi}^n\cdot\dot{\mathbf{j}^n} - \mathbf{\Pi}^p\cdot\dot{\mathbf{j}^p}, \quad \Phi_\rho = \left(\frac{\theta}{\rho}\mathbf{k}\right)\cdot\nabla\rho. \tag{39}$$

The non-negative of the specific heat C follows from the concavity of Ψ with respect to θ. In semiconductor crystals ρ is practically constant, so that the terms $\left(\frac{\theta}{\rho}\mathbf{k}\right)\cdot\nabla\rho$, $\frac{1}{\rho}\dot{\rho}\mathcal{E}_i P_i$ and $\frac{1}{\rho}\dot{\rho}P_i$ can be considered vanishing in equ.s (19), (22), (27), (28), $(39)_2$, in equ.s (14), (15) and in equ. (17), respectively.

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TESTING THE CONVERGENCE HYPOTHESIS THROUGH A SIMPLE ECONOMETRIC MODEL

Carlo Migliardo[1]

1. Introduction

In this paper, we test the convergence hypothesis among a large numbers of countries. Over the last century, many countries have expressed a heterogeneous rate of growth. Some poor States have catched up richer nations of the World. Other economies denote a stagnant path of growth in the long-run. A large body of literature related to cross-country inequality and growth (starting from Solow (1957), Romer (1990) and Lucas (1988)) have been developed to investigate the mysterious different growth path, sometimes, for similar Nations. Even if, several different economic schools of thought challenged to explain and to give the "golden rules" to achieve the welfare of the richer nations, no one could give a definitive answer to the key question about this issue. For instance, a typical contraposition present in the literature, whether growth is due to an exogenous or endogenous knowledge accumulation. From an empirical point of view, Barro and Sala-i-Martin (1999) and Mankiw et. al. (1992), among many others, have employed several econometric models to individuate the casual relation between economic growth and other variables.

The general aim of this research is to explore the possibility to find new empirical regularities in influence on a nation's development performance, building on the relevant past evidence about growth models. Above all, this research applies the simplest econometric model to a several cross-section data for Countries. In particular, we adopt a Generalized Least Squares[2] (GLS henceforth) method to identify the drivers for absolute as well as conditional convergence hypothesis. Since, countless efforts have been focused on this topic on growth theory, becoming an important branch of economics theories. Several applied works have been considered to examine the causality relation between income and several indicators, such as, educational, saving rate, productivity and R&D investment. Overall, many researches[3] argued that R&D investment

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[2] For a description of GLS technique, see Amemiya (1985) and Greene (2000)

may drive business cycle. One confirm comes from the Lisbon 2010 goals; since, it requires, among numerous targets values for economics variables, a three percentage of GDP should be invested to augment the productivity toward R&D investment.
In a first step, this survey implements the Absolute convergence. Then, the model tests an alternative assumption of conditional convergence. Initially, we document the convergence using the whole dataset. Then, our exercise investigates only the growth performance among OECD countries.
The analysis is structured as follow: section 2 shows the absolute and conditionated convergence. Section 3 deals with convergence among OECD club. Section 4 concludes.

2. Test for the absolute convergence hypothesis

In this section, we consider the empirical result for the absolute convergence: so we test if poorer countries growth faster and tend thus to catch-up the richer countries. This hypothesis implies that the growth rate of real per capita GDP (in log) from 1960 to 2000 would tend to be inversely related to the level of real per capita GDP in 1960. The dataset is composed of 104 countries drawn from the PST1 data set selected by Pen World Table.
Y00 is real GDP per worker relative to the US in 2000, G is the GDP growth rate (1960-2000), I is the investment share of real GDP, EDU represents the high skill labour force, N is the growth rate of population.

Dependent Variable: G
Method: Generalized Least Squares
Date: 04/12/07 Time: 16:20
Sample: 1 104
Included observations: 104

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	0.023744	0.008373	2.835631	0.0055
Y60	0.004951	0.024439	0.202589	0.8399
R-squared	0.000402	Mean dependent var		0.024990
Adjusted R-squared	-0.009398	S.D. dependent var		0.057631
S.E. of regression	0.057901	Akaike info criterion		-2.841104
Sum squared resid	0.341963	Schwarz criterion		-2.790250
Log likelihood	149.7374	F-statistic		0.041042
Durbin-Watson stat	1.845563	Prob(F-statistic)		0.839860

Nevertheless, the empirical result of a regression shows that the correlation between per capita Growth Rate and the initial level of income is slightly positive (0.004951). Furthermore, the t-Statistic is smaller than one and the value for the probability is high (84%), so we can accept the hypothesis that true coefficient is zero. Additionally the R-squared is close to zero, and Durbin-Watson stat evidence a little positive autocorrelation. Empirically the coefficient of initial level of GDP (Y60) into the equation represents the rate of convergence. Therefore, we have to reject the absolute convergence hypothesis among these countries.

In order to reconciling the convergence hypothesis with the dataset, we have to rely on the concept of conditional convergence. Besides, we analysis the convergence hypothesis considering also the educational skills (EDU), the ratio of real gross domestic investment to GDP (I), and the medium rate growth of the population (N). Unfortunately, for 28 countries has not available the EDU value. Therefore, we consider 86 nations instead of 104

Dependent Variable: G
Method: Least Squares
Date: 04/18/07 Time: 17:35
Sample(adjusted): 1 86
Included observations: 86 after adjusting endpoints

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	-0.011120	0.016887	-0.658484	0.5121
Y60	-0.119220	0.045668	-2.610612	0.0108
EDU	0.012924	0.003960	3.263870	0.0016
I	-0.000181	0.000402	-0.449273	0.6544
N	0.186051	0.226013	0.823188	0.4128

R-squared	0.116629	Mean dependent var	0.027023
Adjusted R-squared	0.073006	S.D. dependent var	0.062824
S.E. of regression	0.060488	Akaike info criterion	-2.716371
Sum squared resid	0.296360	Schwarz criterion	-2.573676
Log likelihood	121.8039	F-statistic	2.673554
Durbin-Watson stat	2.189701	Prob(F-statistic)	0.037710

Let us comment our GLS regressor coefficients:

Initial Per Capita GDP:

The coefficient for initial per capita GDP is negative confirming the conditional convergence at the rate of 11% percent per year. The t-Statistics is greater than one in magnitude and the Probability (1%) is smaller than 5%, so we can reject the Hypothesis null that the true coefficient is zero at the 1% significance level.

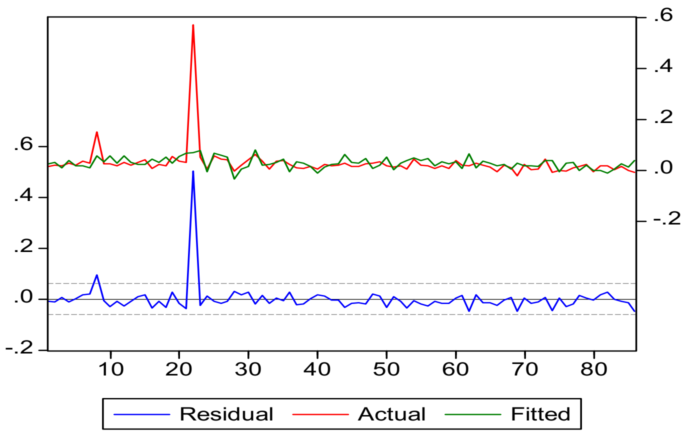
High skilled workforce:

The estimated coefficient is positive but it explains only 1,2% of the dependent variable, the T-Statistic is very high and the hypothesis that educational variable does not enter into the growth equation is rejected with s p-value of 0.0016(0,16%). Hence, the variable is significant to explain the growth rate. Although, Educational skills seems not to be the most important regressor.

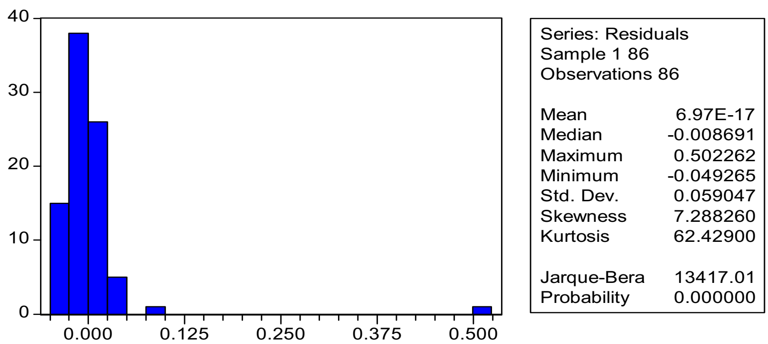
Investment ratio:

The result suggests, paradoxically, that investment influenced negatively growth, given the fitted coefficient for investment is negative, but the t-Statistics is lower than one and the high value for the probability suggest to accept the Hypothesis that the true coefficient is zero.

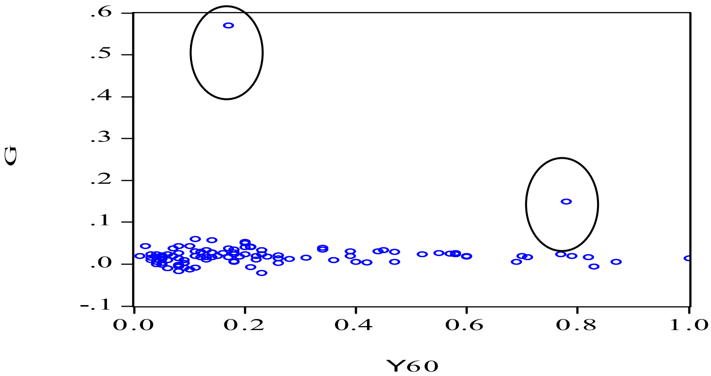
Population growth rate:
The estimated coefficient on the population growth rate is positive (18%), but it is not significant given the high value for the p-value (41%) and the t-Statistic is positive but lower than one (0.82).
A quick glance at the result reveals that the fit has improved respect the previous test. Now we have not any more serial correlation for the residual, given that The Durbin Watson Statistic is around 2 (2.18). Also the f-Statistic is greater than 1 (2.67) with a p-value(3%) less than 5%, so we can rejected the null hypothesis that all slope coefficients are equal to zero. However, the R-Square (0.11) is too low.
If we analysis the graphical representation of the regression, it is clear that regression fits very well. On the other hand, the dataset contains some outlier data.



However, the histogram displays clearly that the frequency distribution of the residuals is not normal distributed:



Scatter diagram evidences the presence of two outliers: Hong Kong with a rate of growth of (57%) and Australia (15%).



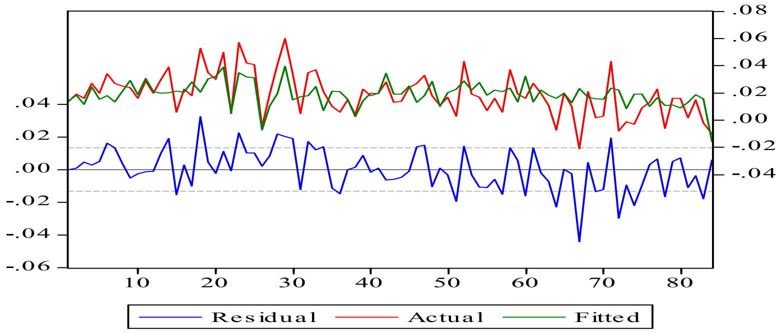
Thus, we get these results by excluding both the outliers:

Dependent Variable: G
Method: Generalized Least Squares
Date: 04/19/07 Time: 11:22
Sample(adjusted): 1 84
Included observations: 84 after adjusting endpoints

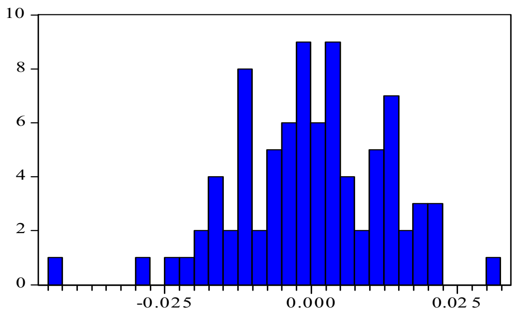
Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	0.012906	0.004052	3.185257	0.0021
Y60	-0.048434	0.010249	-4.725747	0.0000
EDU	0.004302	0.000902	4.771278	0.0000
I	-6.74E-05	8.82E-05	-0.764775	0.4467
N	-0.117150	0.058880	-1.989658	0.0501

R-squared	0.315062	Mean dependent var	0.019095
Adjusted R-squared	0.280381	S.D. dependent var	0.015624
S.E. of regression	0.013254	Akaike info criterion	-5.751365
Sum squared resid	0.013878	Schwarz criterion	-5.606673
Log likelihood	246.5573	F-statistic	9.084721
Durbin-Watson stat	1.882017	Prob(F-statistic)	0.000004

It straightforward to observe that fit has improved. Since, the p-value is not significant, only for the investment ratio. In addition, R-Squared has grown (0.315) and Durbin Watson is still bellowing 2. Examining the coefficients, it is clear that N is the best regressor, while I is not significant to explain Growth. If we look at the bottom up representation for actual-fitted-residual graph, it is clear that regression has improve, but the residual shows the typical heteroschedasticity problem for cross section analysis:



Looking at this histogram, it is easy to observe, that the residuals are "well behaved" .The distribution is symmetric (Skewness close to zero), and Kurtosis is next to 3.



Series: Residuals
Sample 1 84
Observations 84

Mean	3.17E-17
Median	-5.64E-05
Maximum	0.032439
Minimum	-0.044105
Std. Dev.	0.012931
Skewness	-0.358915
Kurtosis	3.697731
Jarque-Bera	3.507382
Probability	0.173134

3. **Considering now the set of OECD countries**

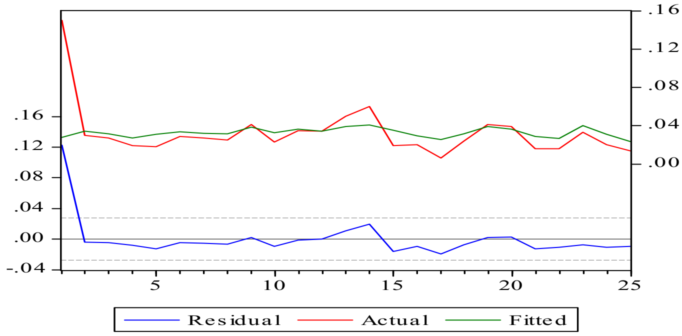
Unluckily for Hungary, Luxembourg, Poland, Slovak Republic, the data are incomplete, thus we have to keep out them.
First, we will test the absolute convergence for these countries club:

Dependent Variable: G
Method: Generalized Least Squares
Date: 04/19/07 Time: 17:22
Sample(adjusted): 1 25
Included observations: 25 after adjusting endpoints

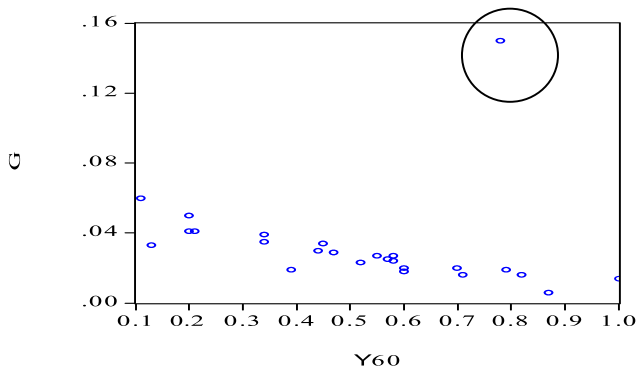
Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	0.042740	0.013236	3.229149	0.0037
Y60	-0.019498	0.023253	-0.838537	0.4104

R-squared	0.029665	Mean dependent var	0.032640
Adjusted R-squared	-0.012524	S.D. dependent var	0.027262
S.E. of regression	0.027433	Akaike info criterion	-4.277550
Sum squared resid	0.017309	Schwarz criterion	-4.180040
Log likelihood	55.46937	F-statistic	0.703144
Durbin-Watson stat	1.074137	Prob(F-statistic)	0.410360

As we expected, we have a negative coefficient for the regressor Y60, however the t-Statistic is smaller than one, the p-value (41%) is too high. Hence, the fitted coefficient is not significant. In addition, the Durbin-Watson stat is very low (1), evidencing a high autocorrelation into the residual.
Taking into consideration the actual –fitted graph, it confirms the presence of some outlier into the regression.



In fact looking at this graph bellow, we have a rate of growth too higher (15%) than other OECD countries.



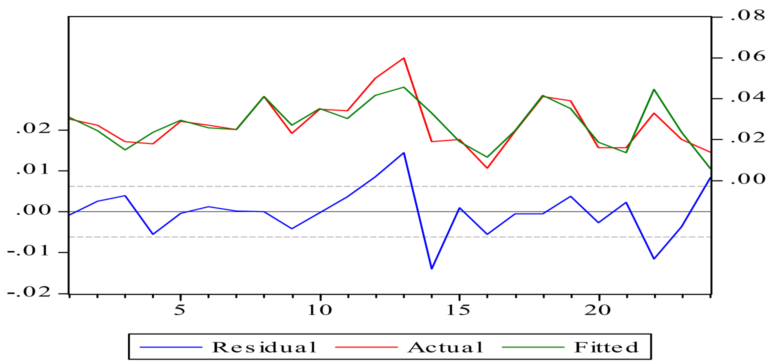
Therefore, if we keep out the outlier (Australia) from the club, we get
Dependent Variable: G
Method: Generalized Least Squares
Date: 04/22/07 Time: 16:21
Sample(adjusted): 1 24
Included observations: 24 after adjusting endpoints

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	0.050474	0.003003	16.80984	0.0000
Y60	-0.044814	0.005374	-8.338416	0.0000

R-squared	0.759640	Mean dependent var	0.027750
Adjusted R-squared	0.748714	S.D. dependent var	0.012319
S.E. of regression	0.006175	Akaike info criterion	-7.256836
Sum squared resid	0.000839	Schwarz criterion	-7.158665

Log likelihood	89.08203	F-statistic	69.52919
Durbin-Watson stat	2.171718	Prob(F-statistic)	0.000000

These results reveal that the fit is very good now: The fitted coefficient is significant given that the t-Statistic (-8.33) is greater than one in magnitude and the p-value is zero. Moreover, we have any more serial correlation into the residual, given that The Durbin Watson Statistic is around 2 (2.18). Furthermore, The R-Square is high (0.76). The following graph shows the goodness of the regression, although the residual shows the presence of heteroskedasticity, typical for a cross section analysis.



Given the negative sign for the coefficient, we can accept the absolute convergence, among the OECD countries at the rate of 4,48% .

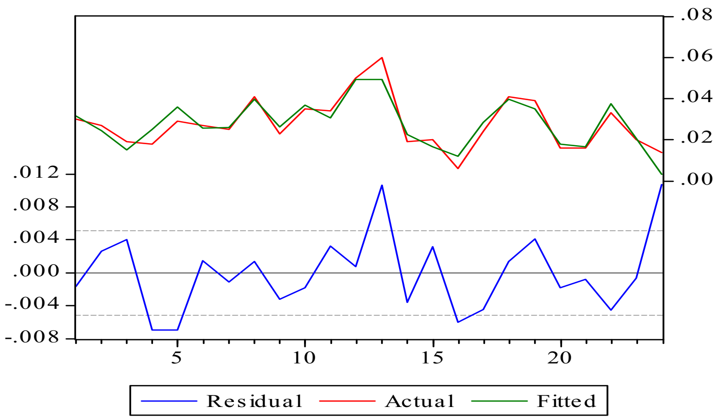
Now we will test the Conditionated Convergence for OECD countries.

Dependent Variable: G
Method: Generalized Least Squares
Date: 04/22/07 Time: 18:59
Sample(adjusted): 1 24
Included observations: 24 after adjusting endpoints

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	0.035022	0.007084	4.944036	0.0001
Y60	-0.050018	0.007735	-6.466221	0.0000
EDU	0.000710	0.000848	0.836367	0.4133
I	0.058655	0.028585	2.051909	0.0542
N	-0.271789	0.199887	-1.359710	0.1898

R-squared	0.855575	Mean dependent var	0.027750
Adjusted R-squared	0.825170	S.D. dependent var	0.012319
S.E. of regression	0.005151	Akaike info criterion	-7.516215
Sum squared resid	0.000504	Schwarz criterion	-7.270787
Log likelihood	95.19458	F-statistic	28.13902
Durbin-Watson stat	1.884023	Prob(F-statistic)	0.000000

It is easy to see that regression has improved, the R-squared growth from 0.76 to 0.85, even EDU is not significant for the regression (t-Statistics is lower than one and probability is too high). Population growth rate seem to be the best regressor (27%). Now as we expected, the coefficient of Investment ratio is positive and significant at 95% confidence level.
We have a positive effect on Growth when the initial level of GDP is low. There is also evidence that higher Investment will speed up convergence.
These empirical results on conditional convergence are consistent with the neoclassical growth model. On the contrary, EDU variable is never significant to explain growth.
The graph below shows the goodness of the regression.



On the contrary, if we test the stability of the coefficient excluding OECD countries,
Chow Breakpoint Test for Absolute Convergence

Chow Breakpoint Test: 80			
F-statistic	12.94479	Probability	0.000010
Log likelihood ratio	23.92795	Probability	0.000006

The outcomes shows that Chow Breakpoint Test for the absolute convergence excluding the two outliers (Austria, Australian) reject null hypothesis that OECD and all countries can be included together. In fact, the calculated F-statistic of 12.99 exceeds the critical F-value of 11.08 for the 5% level of significance so the null hypothesis of no structural change can be rejected. The reported probability is the marginal significance level of the F-test. It supports this result in that rejecting the null hypothesis would be wrong less than 0.0010% of the time.
The calculated value for LR test statistic of 23.92 exceeds of 15.43 for the 5% level of significance and 8.58 for the 1% level of significance so the null hypothesis of no structural change can be rejected.

The reported probability supports this result in that rejecting the null hypothesis would be wrong less than 0.0006% of the time.

Chow Breakpoint Test for the conditioned Convergence.

Chow Breakpoint Test: 80

F-statistic	3.268702	Probability	0.010113
Log likelihood ratio	16.76255	Probability	0.004973

These outcomes are similar than below, so we can once again reject the null hypothesis of no structural change excluding OECD countries from the dataset.

4. Conclusion

We have presented a simple econometric exercise to verify the convergence assumption. The survey, try to answer to several questions. On the one hand, whether poor countries tend to grow faster than industrialized countries. In a Solow model, nations converge to their growth of steady state. On the other hand, there is some evidence that convergence maybe possible only for some samples or "countries club." Hence, convergence process may be occurs or not, depends on the characteristics of the country.

We can conclude outcomes toward generalized least square tools reveal interesting results. First, the important implications and understanding of the process of economic growth and the role of policies in influencing it. Given that human capital evidences a too little positive relation with growth other more important variables should insert into the GLS regression. On the contrary, the absolute convergence assumption can be accepted only for OECD States. The goodness-of-fit of both distributions, assessed through graphical techniques and robust tests, however, is not only to be considered a new result *per se*, since it also leads to a few important implications. Above all, this cross-country analysis confirms the presence of assumption that the long-run growth rate of output per worker is an increasing function of investment ratio, roughly speaking, consistent with the assumption of Neoclassical growth theory. At odds with the many growth models, Human capital investment has very little influences to explain the cross-country income differences. Maybe, this econometric analysis shows the typical cross-section data model issues. In fact, it denotes heteroskedasticity presence in the error terms, some outliers and so on. Perhaps this work should to consider the animal spirit[4] assumption to explain the stylized fact on growth.

[4] J. M. Keynes (1936) argued the idea that waves of spontaneous optimism might drive business cycles "*Most, probably, of our decisions to do something positive, the full consequences of which will be drawn out over many days to come, can only be taken as the result of animal spirits - a spontaneous urge to action rather than inaction, and not as the outcome of a weighted average of quantitative benefits multiplied by quantitative probabilities.*"

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RENDICONTI DEL CIRCOLO MATEMATICO DI PALERMO
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Geographic information and global index of spatial autocorrelation

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Summary

In this paper we try to consider the possibility of utilising the geographic information of the territory, in order to obtain a global index of spatial autocorrelation. Precisely, our aim is to find an index in which is possible to measure the spatial variability of a phenomenon by taking into account some elements of territorial information which are generated from the S-DSMA procedure.

1. Introduction

In statistical analysis regarding territory, the study of problems involving the concept of space is primarily interesting. In fact, phenomena of an economic, social or environmental nature are characterised precisely by spatial dependence, so it is unusual to find independent spatial data. The relationship between the events happening in two defined neighbouring units of territory is usually defined as spatial dependence. According to a classical configuration of a traditional weight system (Moran, Geary *interalia*), there is an association between the measure of the variability of phenomena and the so-called system of interconnection, that is a mechanism able to organise spatial units territorially (contiguity matrix), in the context in which it is realised. The main problem of a traditional interconnection system, as reported by Dacey (and others), is that it is invariant with respect to topological transformations of the territory. In the past, various analysts have tried to solve this problem by introducing a weighting system in order to consider certain physical elements which play a more or less important role in spatial structure. Cases in point are Dacey (1968), who considers the form of the units, Bodson and Peeters (1975), who introduce a general accessibility weight by combing the influence of several channels of communication, and Cliff and Ord (1981) who introduce the measurement of distance and the length of the borders between spatial units. More recently a "General Weight Matrix" has been proposed according the S-DSMA procedure (La Tona L., Mazza A., Mucciardi M., 2006). This method, in fact, has the important property of being "sensitive" with respect to the different territorial configurations as the consequence of the introduction of particular parameters regarding the distance and surface of areal units. By adopting such procedure, the problem of topological invariance is solved. Referring to these observations, in this paper we try to consider the possibility of utilising the geographic information of the territory, in order to obtain a global index of spatial autocorrelation (GISA). Our aim is to find a modification of the R index (Alleva, 1987), in which is possible to measure the spatial variability of a phenomenon by taking into account some elements of territorial information which are generated from the S-DSMA procedure. The paper concludes with an application of a global index.

2. A global index of spatial autocorrelation using the S-DSMA procedure

Before introducing the solution proposed, we should briefly remember that the S-DSMA procedure (with local reweighting) determines the two types of weight matrices: 1) matrix $\mathbf{E}^{\mathrm{k}}$ whose values ε_{ij}^{k} represent the interconnection, in the generic partition k, between the territorial barycentres or centroids of the areal units (the weights generated in the "radius" of interconnection h^{k} are sensitive to the effective distance of each unit); 2) matrix $\mathbf{B}^{\mathrm{k}}$ whose values γ_{ij}^{k}, at the generic partition k, represent the weights in function of the physical characteristics of the areal units. Finally, by introducing a suitable function on the coefficients obtained from the two matrices $\mathbf{E}^{\mathrm{k}}$ and $\mathbf{B}^{\mathrm{k}}$, it is possible to obtain a single matrix $\overline{\mathbf{\Omega}}^{\mathrm{k}}$ (with $\overline{\omega}_{ij}^{k}$ generic element of matrix $\overline{\mathbf{\Omega}}^{\mathrm{k}}$) where, in each territorial partition, factors of "distance" and "surface" in areal units are evaluated simultaneously.

Let $A=\sum_{k=1}^{t}\sum_{i=1}^{n}\sum_{j=1}^{n}\overline{\omega}_{ij}{}^{k}$ the maximum number of links it is possible to obtain from a particular territorial configuration in the maximum interconnection radius h^{t}. In the distance h^{t} all the units are linked to each other.

We define $J_{(k)}$ the accumulated percentage of the total connectivity of the units, in the generic interconnection radius h^{k}, as the following quantity:

$$J_{(k)}=\frac{\sum_{i}^{n}\sum_{j}^{n}\overline{\omega}_{ij}^{k}}{A}\downarrow \quad \text{with k = 1....t} \qquad (J_{(0)}=0 \text{ e } J_{(t)}=1) \qquad (1)$$

In the same way, we define $V_{(k)}$, the accumulated percentage of the variability of the phenomenon X "absorbed" by the linked elements, in the generic interconnection radius h^{k}, as the following quantity :

$$V_{(k)} = \frac{\sum_{i}^{n}\sum_{j}^{n}\left(x_i - x_j\right)^2 \overline{\omega}_{ij}^{k}}{D} \downarrow \text{ with k = 1....t} \qquad (V_{(0)} = 0 \text{ e } V_{(t)} = 1) \qquad (2)$$

with

$$D = \sum_{k=1}^{t}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(x_i - x_j\right)^2 \overline{\omega}_{ij}^{k}$$

The differences $J_{(k)} - J_{(k-1)}$ e $V_{(k)} - V_{(k-1)}$ represent respectively the contribution in terms of "connectivity" and "variability" in the generic distance h^k. In a system of spatial incorrelation the accumulated percentage of variability $V_{(k)}$ should not differ from the accumulated percentage of connectivity $J_{(k)}$. In other words, if the relative contributions in terms of variability and connectivity increase proportionally to the variation of interconnection radius, we will graphically obtain a straight line at a perfect angle of 45°.

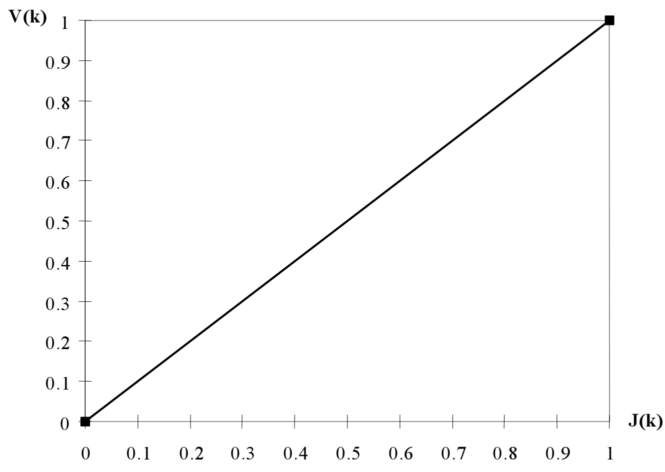


Fig 1. – Curve of spatial autocorrelation (case of incorrelation).

Meanwhile, in the case of positive (negative) autocorrelation, the contribution in terms of variability will be smaller (larger) than the contribution in terms of connectivity in the generic interconnection radius h^k. This situation may be assessed by considering the tangent of the angle formed by the straight line with the x-axis:

$$tg^k(\alpha) = \frac{V_{(k)} - V_{(k-1)}}{J_{(k)} - J_{(k-1)}} \tag{3}$$

with,

$tg^k(\alpha) < 1$ positive autocorrelation,

$tg^k(\alpha) > 1$ negative autocorrelation,

$tg^k(\alpha) = 1$ no autocorrelation.

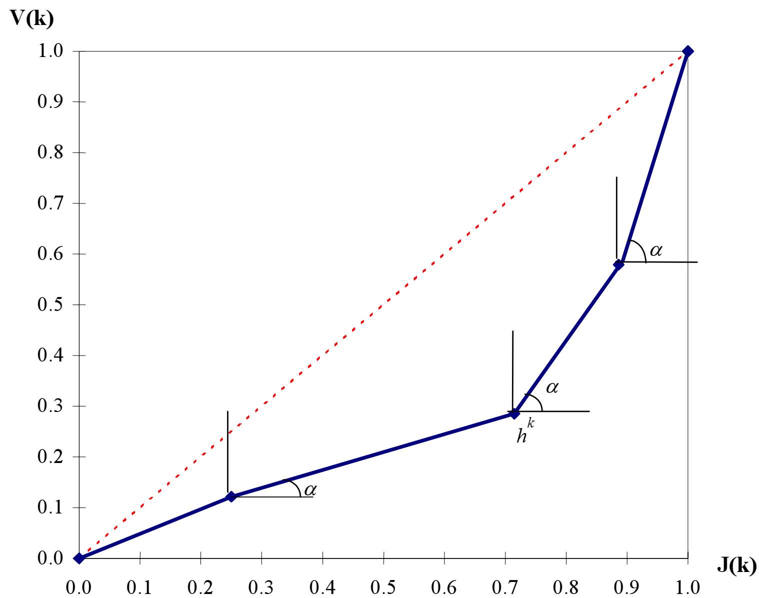


Fig 2. – Values assumed by the angle α with varying values of h^k .

Overall, the spatial autocorrelation may be assessed through the global index of spatial autocorrelation (GISA) defined by :

$$GISA = 1 - 0.5\sum_{k=1}^{t}\left(V_{(k)} + V_{(k-1)}\right)\left(J_{(k)} - J_{(k-1)}\right) \tag{4}$$

where $0.5\left(V_{(k)} + V_{(k-1)}\right)\left(J_{(k)} - J_{(k-1)}\right)$ represents the area of the kth trapezium (see figure 3).

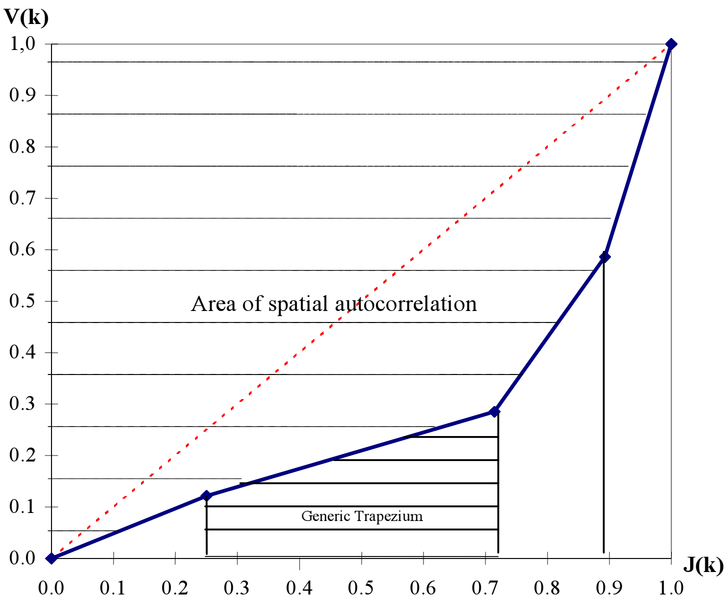


Fig 3. - Area of spatial autocorrelation.

3. Application of the global index of autocorrelation to the quality of the rail service perceived by the user.

The application proposed provides for the calculation of the global index of spatial autocorrelation (GISA)[1] by considering, for each region of Italy, seven indicators of customer satisfaction (Istat, 2004)[2]. The indicators used regard the percentage of people of 14 years and over who use the train and say they are satisfied with regards to: “frequency of trains” (Fre), “punctuality” (Pun), “chances of finding a seat” (Fis), “cleanliness of the carriages” (Cle), “convenience of train times” (Ttm), “cost of the ticket” (Cst) and “service information” (Inf). As far as regards geographical information for the territory, the calculation is made by considering a matrix of distance between the regional territorial barycentres (km) and a vector of overall surface area (hectares) for each region. The output generated for each spatial order includes: 1) the interconnection distance generated by the procedure S-DSMA; 2) the total number of links; 3) the variability produced by the linked elements; 4) the accumulated percentage of territorial interconnection $J_{(k)}$; 5) the accumulated percentage of variability produced by the connected elements $V_{(k)}$; 6) the angle

[1] Calculations performed using S-Joint software (Mucciardi-Bertuccelli, 2007).
[2] Compared to the data provided by Istat, the independent provinces of Trento and Bolzano were excluded from the analysis.

of the straight line with the x-axis; 7) the GISA index (figures 4-10). From the analysis of the results (fig. 11) it can be observed that all the indicators display positive spatial autocorrelation in the first interconnection radius (376 km). Specifically, the indicators "finding a seat" and "frequency of trains" are those which show a higher percentage of satisfied customers, with GISA values of 0.60 and 0.56 respectively. We find the opposite, meanwhile, for the indicators "cost of the ticket" and "cleanliness of carriages", with satisfaction percentages of 35% and 32% respectively. In this case, especially for the indicator "cost of the ticket", the percentage value of the indicator is poorly correlated with the territory (GISA=0.55). Lastly, by projecting the various pairs of coordinates formed by the satisfaction percentage (x-axis) and the value of the GISA index (y-axis), we obtain the "spatial mapping" of satisfaction (see graph 1). The graph shown provides us with a simultaneous vision of values of variables and correlation with the territory.

Spatial order (k)	h (km)	Joints	Variability	J(k)	V(k)	Angle (degrees)	GISA
1st	376.92	164	19.157.91	43.16%	37.62%	41.08	0.556
2nd	470.70	54	5.619.62	57.37%	48.65%	37.83	
3rd	539.71	40	5.441.53	67.89%	59.34%	45.43	
4th	653.56	44	5.418.58	79.47%	69.98%	42.58	
5th	750.97	32	4.929.20	87.89%	79.66%	48.98	
6th	861.18	20	4.901.00	93.16%	89.28%	61.33	
7th	962.95	16	3.558.37	97.37%	96.27%	58.93	
8th	1043.39	8	1.851.11	99.47%	99.90%	59.92	
9th	1064.77	2	50.00	100.00%	100.00%	10.57	

Fig. 4 – Output of the variable "frequency of trains"

Spatial order (k)	h (km)	Joints	Variability	J(k)	V(k)	Angle (degrees)	GISA
1st	376.92	164	18,355.95	43.16%	33.31%	37.66	0.576
2nd	470.70	54	7,772.06	57.37%	47.42%	44.78	
3rd	539.71	40	6,399.78	67.89%	59.03%	47.81	
4th	653.56	44	5,478.22	79.47%	68.97%	40.65	
5th	750.97	32	3,585.96	87.89%	75.48%	37.70	
6th	861.18	20	4,787.77	93.16%	84.17%	58.79	
7th	962.95	16	5,535.12	97.37%	94.21%	67.26	
8th	1043.39	8	2,956.30	99.47%	99.58%	68.57	
9th	1064.77	2	233.28	100.00%	100.00%	38.81	

Fig 5 – Output of the variable "punctuality"

Spatial order (k)	h (km)	Joints	Variability	J(k)	V(k)	Angle (degrees)	GISA
1st	376.92	164	9,775.49	43.16%	28.50%	33.44	0.596
2nd	470.70	54	4,121.87	57.37%	40.52%	40.22	
3rd	539.71	40	4,293.17	67.89%	53.04%	49.94	
4th	653.56	44	6,438.13	79.47%	71.81%	58.33	
5th	750.97	32	2,551.91	87.89%	79.26%	41.46	
6th	861.18	20	2,912.61	93.16%	87.75%	58.21	
7th	962.95	16	2,452.59	97.37%	94.90%	59.51	
8th	1043.39	8	1,410.87	99.47%	99.01%	62.90	
9th	1064.77	2	338.00	100.00%	100.00%	61.90	

Fig 6 – Output of the variable "chances of finding a seat"

Spatial order (k)	h (km)	Joints	Variability	J(k)	V(k)	Angle (degrees)	GISA
1st	376.92	164	10,630.35	43.16%	27.88%	32.86	0.567
2nd	470.70	54	7,043.39	57.37%	46.35%	52.43	
3rd	539.71	40	4,999.33	67.89%	59.46%	51.24	
4th	653.56	44	7,751.03	79.47%	79.79%	60.33	
5th	750.97	32	3,549.97	87.89%	89.10%	47.87	
6th	861.18	20	1,976.97	93.16%	94.29%	44.57	
7th	962.95	16	1,466.53	97.37%	98.13%	42.41	
8th	1043.39	8	658.13	99.47%	99.86%	39.35	
9th	1064.77	2	54.08	100.00%	100.00%	15.08	

Fig 7 – Output of the variable “cleanliness of the carriages”

Spatial order (k)	h (km)	Joints	Variability	J(k)	V(k)	Angle (degrees)	GISA
1st	376.92	164	18,041.80	43.16%	31.52%	36.15	0.598
2nd	470.70	54	6,136.27	57.37%	42.25%	37.04	
3rd	539.71	40	6,875.96	67.89%	54.26%	48.78	
4th	653.56	44	6,171.47	79.47%	65.05%	42.96	
5th	750.97	32	4,985.36	87.89%	73.76%	45.97	
6th	861.18	20	5,622.08	93.16%	83.58%	61.82	
7th	962.95	16	5,483.69	97.37%	93.16%	66.28	
8th	1043.39	8	3,543.69	99.47%	99.35%	71.22	
9th	1064.77	2	369.92	100.00%	100.00%	50.85	

Fig 8 – Output of the variable “convenience of train times”

Spatial order (k)	h (km)	Joints	Variability	J(k)	V(k)	Angle (degrees)	GISA
1st	376.92	164	8,189.14	43.16%	35.51%	39.45	0.550
2nd	470.70	54	3,934.02	57.37%	52.57%	50.21	
3rd	539.71	40	2,451.38	67.89%	63.20%	45.28	
4th	653.56	44	2,258.77	79.47%	73.00%	40.23	
5th	750.97	32	1,709.84	87.89%	80.41%	41.36	
6th	861.18	20	883.36	93.16%	84.24%	36.05	
7th	962.95	16	1,514.83	97.37%	90.81%	57.34	
8th	1043.39	8	1,381.63	99.47%	96.80%	70.64	
9th	1064.77	2	737.28	100.00%	100.00%	80.65	

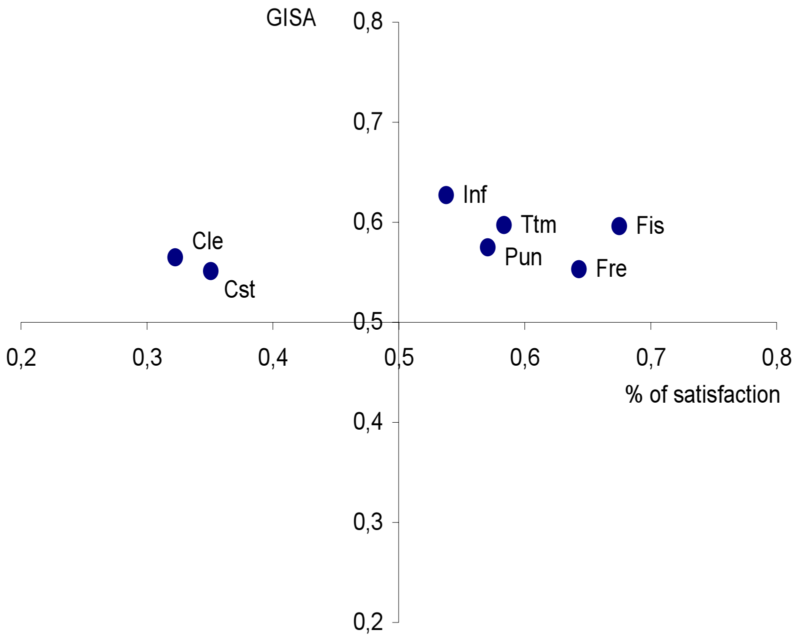
Fig 9 – Output of the variable “cost of the ticket”

Spatial order (k)	h (km)	Joints	Variability	J(k)	V(k)	Angle (degrees)	GISA
1st	376.92	164	7,856.57	43.16%	26.32%	31.38	0.627
2nd	470.70	54	3,747.71	57.37%	38.88%	41.46	
3rd	539.71	40	3,300.51	67.89%	49.94%	46.41	
4th	653.56	44	3,009.35	79.47%	60.02%	41.05	
5th	750.97	32	3,744.73	87.89%	72.57%	56.13	
6th	861.18	20	3,303.67	93.16%	83.63%	64.57	
7th	962.95	16	3,072.64	97.37%	93.93%	67.76	
8th	1043.39	8	1,587.54	99.47%	99.25%	68.41	
9th	1064.77	2	224.72	100.00%	100.00%	55.04	

Fig 10 – Output of the variable “service information”

Index of customer satisfaction	% of satisfaction (Italy) [3]	GISA	Angle (degree) 1st	Angle (degree) 2nd	Angle (degree) 3rd	Angle (degree) 4th	Angle (degree) 5th	Angle (degree) 6th	Angle (degree) 7th	Angle (degree) 8th	Angle (degree) 9th
Finding a seat	67.52	0.60	33.44	40.22	49.94	58.33	41.46	58.21	59.51	62.90	61.90
Frequency of trains	64.31	0.56	41.08	37.83	45.43	42.58	48.98	61.33	58.93	59.92	10.57
Train times	58.36	0.60	36.15	37.04	48.78	42.96	45.97	61.82	66.28	71.22	50.85
Punctuality	57.08	0.58	37.66	44.78	47.81	40.65	37.70	58.79	67.26	68.57	38.81
Information	53.78	0.63	31.38	41.46	46.41	41.05	56.13	64.57	67.76	68.41	55.04
Cost of ticket	35.09	0.55	39.45	50.21	45.28	40.23	41.36	36.05	57.34	70.64	80.65
Cleanliness	32.25	0.57	32.86	52.43	51.24	60.33	47.87	44.57	42.41	39.35	15.08

Fig 11 – Summary of output of the indicators (the angles < 40° in dark grey; the angles >= 40° and < 45° in grey)



Graph. 1 – Spatial mapping of the customer satisfaction indicators

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COUNTABLE MODELS AND INFINITE TREES

MARIAFORTUNA PARATORE - GAETANA RESTUCCIA

Abstract. The property of the countable model is introduced for a set U of formulas in the predicate calculus. Algebraic tools are applied to study (finite or infinite) semantic trees associated to a formula of the propositional or predicate calculus.

Classification AMS: 03Bxx, 03Cxx, 05Cxx

Introduction

In the predicate calculus given a set U of formulas one defines model an interpretation $\mathcal{I}$ such that the value of A, $v(A)$ is V (true) in $\mathcal{I}$, for each A in U.

The Herbrand model is classical and it is a valid model for any interpretation. It can be a finite or an infinite model. A result affirms that if a set U of formulas has a model, then has a countable model. This follows by construction of a tableau of U, by removing all the existential quantifiers and applying them the δ-rules. In this work one studies the property of countable model for a set U of formulas and one proves the following theorem:
If U is a set of formulas that has the countable property, then U is satisfactorily if and only if it has a countable Herbrand model.
The aim of N.2 is to study the semantic trees associated to a formula of the propositional or predicate calculus from an algebraic point of view, by employing the algebraic graph theory ([4]).

1. Models

It is known that if S is a set of clauses (i.e. if the formula A is in a clausale form), there exists always a set of canonical interpretations such that if S has a model, then it has one of this form.
Such set of interpretations, H_S, is called **Herbrand universe** of S.

We will now give some definitions and theorems that will be used subsequently.

Definition 1.1. *Let S be a set of clauses, $\mathcal{A}$ the set of the constant symbols in S and $\mathcal{F}$ the set of the function symbols of S. One defines Herbrand universe of S, H_S, the set defined by induction:*
1) $\forall a_i \in \mathcal{A}$, $a_i \in H_S$
2) $f_i(t_1, \ldots, t_n) \in H_S, \forall f_i \in \mathcal{F}$ *e* $t_j \in H_S$, $1 \leq j \leq n$.

Remark 1.1. *If the set of the constants of $\mathcal{A}$ is empty, the previous definition begins with an arbitrary constant symbol a.*

Remark 1.2. *If $\mathcal{F} \neq 0$, then H_S is an infinite set.*

Example 1.1. $S = \{p(a) \vee q(x), \neg p(x) \vee \neg p(b) \vee q(x)\}$. *There are two constant symbols and there is not function, therefore*

$$H_S = \{a, b\}$$

Example 1.2. $S = \{\neg p(x, f(x)), p(z, g(z))\} = \{\forall x \neg p(x, f(x)), \forall z p(z, g(z)\}$. *There are not constant symbols, hence we begin with a constant a, but there are two functions f e g:*
$H_S = \{a, f(a), g(a), f(f(a)), f(g(a)), g(f(a)), g(g(a)), ...\}$, *where f and g are unary (1-ary) functions.*

Example 1.3. $S = \{af(x, y) \vee pbf(x, y)\}$,
$H_S = \{a, b, f(a, a), f(b, b), f(b, a), f(a, b), f(a, f(a, a)), f(b, f(a, a)), ...\}$, *being f a 2-ary function.*
S arises from the formula:
$A := \forall x \forall y \exists z p(a, z) \vee \forall x \forall y \exists t p(b, t)$.

Definition 1.2. *A term whose variables are the elements of Herbrand universe is called* ***bases term****.*

Definition 1.3. *A* ***base atom*** *is an atomic formula in which the terms are in the Herbrand universe.*

Definition 1.4. *A* ***base clause*** *is a clause in which the terms are in the Herbrand universe.*

Definition 1.5. *An Herbrand interpretation is an interpretation $w : H_S \to H_S$, such that*

$$w(a) = a,\ \forall a \in H_S,\ a \text{ a constant symbol}$$
$$w(f(t_1, \ldots, t_n)) = f(w(t_1), \ldots, w(t_n)).$$

There is not any restriction as regards the assignation of universe Herbrand relations to the predicates.

Definition 1.6. *Let S be a set of clauses and H_S the Herbrand universe of S. One defines* ***Herbrand model*** *for S an Herbrand interpretation such that satisfies S. More precisely, it is the subset of the Herbrand base for which* $v(p(t_1, \ldots, t_n)) = V$

Definition 1.7. *Let S be a set of clauses and H_S the Herbrand universe of S, the **Herbrand base** B_S , is the set of the closed atoms, which can be formed by predicates in S and terms in H_S.*

Theorem 1.1. *Let S be a set of clauses. S has a model $\Leftrightarrow$ has an Herbrand model.*

Proof.(see [1]).

Definition 1.8. *A set of formulas U has the property of the finite model if U is satisfactorily $\Leftrightarrow$ is satisfactorily in an interpretation whose model is finite.*

Theorem 1.2. *Let U be a set of formulas of the form:*
$\exists x_1 \dots \exists x_k \forall y_1 \dots \forall y_l A(x_1, \dots, x_k; y_1, \dots y_l)$ *where A don't have quantifiers or function symbols. Then U has the property of finite model.*

Proof. ([2], teor. 3.10.2) We construct a tableau for U. We remove the existential quantifiers using the δ-rules and we obtain k constants for A. The tableau is finite as soon as all the substitutions for the universal quantifiers which use these constants will be made:

$$\begin{gathered}\exists x_1 \dots \exists x_k \forall y_1 \dots \forall y_l A(x_1, \dots, x_k; y_1, \dots y_l),\ (l > k)\\ \downarrow \\ \forall y_1 \dots \forall y_l A(a_1, \dots, a_k; y_1, \dots y_l) \\ \downarrow \\ A(a_1, \dots, a_k; \underbrace{a_1, \dots, a_k, \dots, a_1, \dots, a_s)}_{l \quad constants}.\end{gathered}$$

The case $l < k$ is the analogue.
The finite domain is the union of finite domains of this type.

Theorem 1.3. *(*[2]*, teor.3.10.3) If a formula A is satisfactorily, then is satisfactorily in a countable domain.*

Proof. We suppose that the tableau T for A is not closed. Then there exist an open branch such that U, the union set of the labels of the knots b, forms a set of Hintikka. But every set of Hintikka has a model $\mathcal{I}$ for U and $A \in U$. So $\mathcal{I} \models A$.

The definition we introduce now is justified by the fact that formulas which do not have finite models exist.

Definition 1.9. *A set of formulas U has the property of countable model if:*

1) U is satisfactorily $\Leftrightarrow$ is satisfactorily in a countable interpretation, an interpretation whose domain is a countable set.

2) U is not satisfactorily in a finite interpretation.

Example 1.4. *$U = \{A\}$, where $A = A_1 \wedge A_2 \wedge A_3$, has the property of countable model, with*
$A_1 = \forall x \exists y p(x, y)$, $A_2 = \forall x \neg p(x, x)$, $A_3 = \forall x \forall y \forall z(p(x, y) \wedge p(y, z) \Rightarrow p(x, z))$.

Proof. (cf.[2])

Example 1.5. *If $<$ is a total order on the n-uples $(a_1, \ldots, a_n) \in N^n$. For example, we can consider the lexicographic term order, the reverse lexicographic order, the degree lexicographic order, the degree reverse lexicographic order. Then $(N^n, <)$ is a model for A.*
For the definitions of these total orders, see ([1], Chap 1)

Theorem 1.4. *Let U be a set of formulas such that U has the property of the countable model. Then $\exists k > l$ such that U is a set of formulas of the following form:*
$\exists x_1 \ldots \exists x_{k-1} \forall x_k \forall y_1 \ldots \forall y_l \exists y_k A(x_1, \ldots, x_{k-1}, x_k, y_1, \ldots, y_l, y_k)$, *where A don't have quantifiers or function symbols.*

Proof. We suppose that $k > l$ with the previous property. Then the formula U has the following form:
$\exists x_1 \ldots \exists x_{k-1} \forall y_1 \ldots \forall y_l A(x_1, \ldots, x_{k-1}, y_1, \ldots, y_l)$, where A don't have quantifiers or function symbols. Hence A has the property of the finite model. That contradicts the hypothesis.

Definition 1.10. *An Herbrand interpretation is called countable if the domain is a countable Herbrand universe.*

Example 1.6. *$H_S = \{a, f(a), f(f(a)), f(f(f(a))), \ldots\}$ is a countable Herbrand universe, by the map* $N \longrightarrow H_S$

$$i \longrightarrow f(\underbrace{f(f(...(a)))}_{i-times} \in H_S$$

Definition 1.11. *An Herbrand model for a set of clauses S is called countable if the Herbrand interpretation that satisfies S is a set of the countable base Herbrand.*

Theorem 1.5. *Let U be a set of formulas that has the countable property. Then U is satisfactorily $\Leftrightarrow$ has a countable Herbrand model.*

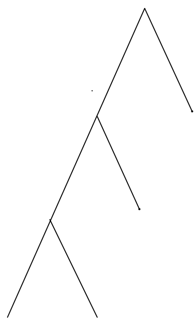
Proof ($\Leftarrow$) It is clear.
($\Rightarrow$) If U has the countable property, then U has the form of the theorem 1.4 and it must contains exactly one Skolem function. From this function we obtain a countable Herbrand model.

Example 1.7. *Let $A_1 := \forall x \exists y p(x,y)$, then $A = A_1 \wedge A_2 \wedge A_3$ has the form of the theorem 1.4. From the first formula A_1, we obtain one Skolem function $f(x)$ and the Herbrand universe $\{a, f(a), f(f(a)), ...\}$.*

2. Semantic trees

The propositional and predicate calculus give new examples of graphs in the usual definitions. More precisely, the graphs are special class of graphs, as we are going to study.

Proposition 2.1. *Any formula A of the propositional calculus is a finite graph that is a tree.*



Proposition 2.2. *A set U of formulas of the propositional calculus is a forest.*

The following definitions need to find properties of our graphs.

Definition 2.1. *Let G be a graph on a vertex set V. A subset $A \subset V$ is a minimal vertex cover for G if:*

a) every edge of G is incident with one vertex in A;
b) there is no proper subset of A with the property a).

Definition 2.2. *A set of vertices of G is independent if no two of them are adjacent.*

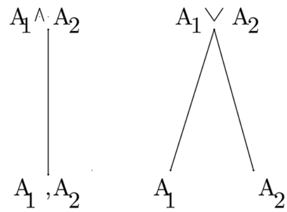
Definition 2.3. *The smallest number of vertices in any minimal vertex cover is called* ***vertex covering number****, denoted $\alpha_0(G)$.*

Definition 2.4. *The independence number of a graph G, $\beta_1(G)$, is the maximum number of independent edges in G.*

Theorem 2.1. *Let A be a formula of the propositional calculus and let G be the associated graph to its semantic tree. Then*

1) $\beta_1(G) = \alpha_0(G)$
2) $\alpha_0(G) = \beta_1(G) \leq 2$

Proof. 1) G is a tree, then G is bipartite. The thesis follows by theorem 6.1.7 of [4].
2) It follows by δ-rules ([2], Chap.2):



Proposition 2.3. *Let A be a formula of the propositional calculus, let G be the associated graph to its semantic tree and $I(G)$ its edge ideal. Then $htI(G) \leq 2$.*

Proof For the corollary 6.1.18 of [4], $htI(G) = \alpha_0(G)$. The thesis follows by the theorem 2.1.

Proposition 2.4. *Let A be a formula of the predicates calculus with the property of countable model . Then we have* 1) *and* 2) *of the theorem 2.1.*

Proof. The same as the propositional calculus.

Definition 2.5. *Let T be a forest, with $\gamma(T)$ we denote the family of its minimal vertex covers. We call the integer $\mid \gamma(T) \mid$ the* ***cover complexity*** *of T.*

Problems:
1) G is Cohen Macaulay ? (cf.[4], 6.1.15)
2) What is the cover complexity of the forest associated to a set U of formulas?

Remark 2.1. *Even if the tree is infinite, the definitions of independence and minimal cover are also valid. We don't may consider the techniques used in* [4] *and by other authors that permit to associate to a polynomial ring, unless we consider a not noetherian ring.*

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POLARIZATIONS OF PLANAR GRAPHS AND GROEBNER BASIS

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Abstract. Graphs represent a geometric model to solve practical problems of connection. They have some applications in the field of the transport and nets of telecommunications. In particular, we study planar bipartite graphs and their associated monomial algebra. Since the graph G is embedded in the plane and divided it into regions, we consider cycles that bound these regions and we study the polarization of G linked to polarizations of each cycle. The algebraic techniques of theory of Groebner bases are used.
Classification AMS: 05Cxx, 13P10

Introduction

The graphs represent a general instrument of analysis of the structure phenomena, the city and territorial analysis uses mostly planar graphs. The street net can be a planar graph: a public square it could be considered a vertex of a graph and the edges are the connection with other roads. Similarly,we can construct a planar graph considering the lines of the city bus.

Let G be a graph with vertex set $V(G) = \{v_1, \dots, v_n\}$ and a collection $E(G)$ of subsets of V that consists of pairs $\{v_i, v_j\}$, for some $v_i, v_j \in V$ called *edges*. $w = \{v_0, z_1, v_1, \dots, z_n, v_n\}$, where $z_i = \{v_{i-1}, v_i\}$ is the edge joining v_{i-1} e v_i is a walk of length n. If $v_0 = v_n$ the walk w is called a closed walk. A cycle is a closed walk with all vertices distinct.

A graph G is bipartite if its vertex set V can be partitioned into disjoint subsets V_1 and V_2, and any edge joins a vertex of V_1 with a vertex of V_2. A graph G is bipartite iff all the cycles of G are even. A graph G is a complete bipartite if all the vertices of V_1 are joined to all the vertices of V_2.

In this work, we study bipartite planar graphs and two objects associated to G:
1. The edge ideal $I(G)$, $I(G) \subset K[X_1, \dots, X_n]$
2. The associated monomial algebra $K[G]$.

Let K be a field, $R = K[X_1, \dots, X_n]$ a polynomial ring and $K[G]$ the K-subalgebra of R. In [2] $K[G]$ is studied via its presentation ideal, $P(G)$. Let $f_1, \dots, f_q$ be the generators of $K[G]$, $P[G]$ is the kernel of the map

$K[T_1, \ldots, T_q] \longrightarrow R$ induced by $T_i \mapsto f_i$. If G is bipartite the generators of $P[G]$ correspond to cycles in G. Since the graph G is embedded in the plane and divided it into regions, we consider cycles that bound these regions and polarizations of G linked to polarizations of each cycle. In particular, we can have term orders that arise from a polarization.
The aim of this paper is to compute the Castelnuovo - Munford regularity of $I(G)$ and $P(G)$ on two particular classes of planar graphs: St_r and $B_{2r'}$. We believe that there exists a bound for both the regularities in terms of the number r of regions of G and we are working in this direction. The techniques of the theory of Groebner Basis are used.

1. Planar graphs

Let G be a graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and a collection $E(G)$ of subsets of V, that consists of pairs $\{v_i, v_j\}$, for some $v_i, v_j \in V$ called *edges*.

Definition 1.1. *A walk of length n in G is alternating sequence of vertices and edges* $w = \{v_0, z_1, v_1, \ldots, z_{i-1}, v_{i-1}, \ldots, z_n, v_n\}$*, where* $z_i = \{v_{i-1}, v_i\}$ *is the edge joining* v_{i-1} *e* v_i*. If* $v_0 = v_n$ *the walk w is called a closed walk. A path is a walk with all its vertices distinct.*

Definition 1.2. *A graph G is connected if for every pair of vertices* v_1 *and* v_2 *there is a path from* v_1 *to* v_2*.*
If $G = \bigcup_{i=1}^{r} G_i$*, where* $G_1, \ldots, G_r$ *are the maximal (w.r.t. inclusion) connected subgraphs of G, then the* G_i *are called connected components of G.*

Definition 1.3. *A cutpoint of a graph G is a vertex whose removal increases the number of connected components.*

Definition 1.4. *A cycle of length n is a closed walk* $\{v_0, v_1, \ldots, v_n\}$ *in which* $n \geq 3$ *and all vertices are distinct. A cycle is even (odd) if its length is even (odd).*

Definition 1.5. *A graph G is bipartite if its vertex set V can be partitioned into disjoint subsets* $V_1 = \{x_1, \ldots, x_n\}$ *and* $V_2 = \{y_1, \ldots, y_m\}$*, and any edge joins a vertex of* V_1 *with a vertex of* V_2*.*

Proposition 1.1. [3] *A graph G is bipartite if and only if all the cycles of G are even.*

Definition 1.6. *A graph G is a complete bipartite if all the vertices of* V_1 *are joined to all the vertices of* V_2*. If* V_1 *and* V_2 *have n and m vertices respectively, we denote such complete bipartite graph by* $K_{n,m}$

Definition 1.7. *A graph is said planar if it can be embedded in the plane and its edges are incident only in the common vertices.*

The graphs K_5 and $K_{3,3}$ are non planar.

This graphs are called Kuratowski's graphs and in the **Kuratowski's theorem** one has that:
A graph is planar if and only if it has no subgraph homeomorphic to K_5 or$K_{3,3}$.

Let G be a bipartite planar graph on vertices $x_1, \ldots, x_n$ and $R = K[X_1, \ldots, X_n]$ a polynomial ring over a field K,with one variable X_i for each vertex x_i.

The *edge ideal* $I(G)$ associated to a graph G is the ideal of R generated by monomials of degree two, X_iX_j, on the $X_1, \ldots, X_n$ variables, such that $\{x_i, x_j\} \in E(G)$ for all $1 \leq i \leq j \leq n$:

$$I(G) = (\{X_iX_j | \{x_i, x_j\} \in E(G)\}).$$

Let $K[G]$ be the K-subalgebra of R generated by $\{X_iX_j | \{x_i, x_j\} \in E(G)\}$, that is called edge subring of G:

$$K[G] = K[\{X_iX_j | \{x_i, x_j\} \in E(G)\}] \subset R.$$

If $f_1, \ldots, f_q$ be the edge generators of G, that are the monomials that correspond to the edges of G, then the presentation ideal $P(G)$ of the $K(G)$ is the kernel of the homomorphism of K-algebras

$$\varphi : B = K[T_1, T_2, \cdots, T_q] \longrightarrow K[G], \qquad T_i \longmapsto f_i.$$

Definition 1.8. *Let $w = \{x_1, \ldots, x_r = x_1\}$ be an even closed walk in G such that $f_i = X_{i-1}X_i$. As*

$$f_1f_3 \cdots f_{r-1} = f_2f_4 \cdots f_r,$$

the binomial

$$T_w = T_1 \cdots T_{r-1} - T_2 \cdots T_r$$

is in $P(G)$. T_w is the binomial associated to w.

Now, we identify a set of generators for $P(G)$ that correspond to the even closed walks of G.

Definition 1.9. *A closed walk of even length is called a monomial walk.*

Now, we give some basic facts on Groebner basis and toric ideal.
Let I be an ideal of R, f a non zero polynomial of R. The initial ideal of I is given by: $in(I) = (\{lt(f) \in I\})$.

Definition 1.10. *Let $I \neq 0$ be an ideal of R and $F = \{f_1, \ldots, f_r\}$ be a subset of I. The set$F$ is called a Groebner basis of I if $in(I) = (lt(f_1), \ldots, lt(f_r))$.*
F is a reduced Groebner basis for I if
i) $lc(f_i) = 1$, $\forall i$
ii)none of the terms occurring in f_i belongs to $in(F\ \{f_i\})$, $\forall i$.

The following result are known:

Proposition 1.2. *The toric ideal $P(G)$ of a edge subring $K[G]$ has a Groebner basis consisting of binomials w.r.t. any monomials ordering of the polynomial ring $K[T_1,\dots,T_q]$*

Proposition 1.3. *If G is a bipartite graph and $P(G)$ the toric ideal of the edge subring $K[G]$, then $P(G)=(\{T_w : w \text{ is an even cycle}\})$*

Proof. See [2].

Proposition 1.4. *If G is a bipartite graph and $P(G)$ the toric ideal of $K[G]$), then the set*

$$\{T_w : w \text{ is an even cycle}\}$$

is a universal Groebner basis of $P(G)$.

Polarizations

Let G be a bipartite planar graph with vertex set $V(G)$ and $R = k[X_1,\dots,X_n]$ a polynomial ring over K.
Fix an embedding of the graph G in the plane. Let c be an even closed walk, $c = T_{i_1},\dots,T_{i_{2r}}$.

For any such walk, write

$$T_c = T_{i_1}T_{i_3}\cdots T_{i_{2r-1}} - T_{i_2}T_{i_4}\cdots T_{i_{2r}}.$$

By previous proposition we have that the generators of $P(G)$ correspond to cycles in G:

$$P(G) = \{T_c | \text{c is a cycle in} G\},$$

where T_c are binomials.

Since G is embedded in the plane, G divides the plane into regions.

Definition 1.11. *If c is a closed walk that bounds a region (not including the unbounded region) c is called atomic cycle.*

Definition 1.12. *$A(G) \subset P(G)$ is the ideal generated by*

$$\{T_c|\ c \text{ is an atomic cycle in} G\}$$

As G is embedded in the plane every cycle in G has a locus in the plane.

In the sequel we will consider graphs connected with no cutvertices (2-connected) and with a fixed embedding in the plane, where n are vertices, r are regions and q the edges.

Definition 1.13. *Let c be an even closed walk of G, $c = T_{i_1},\dots T_{i_t}$. A polarization of c is the set*

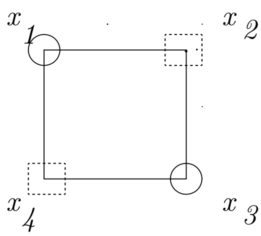
$$\tilde{c} = \{T_{i_1}, T_{i_3},\dots,T_{i_{r-1}}\} \qquad or \qquad \tilde{\tilde{c}} = \{T_{i_2}, T_{i_4},\dots,T_{i_t}\}.$$

Let $\{c_1,\dots,c_r\}$ be the atomic cycles of G. A polarization $\mathcal{P}$ of G is a set $\mathcal{P} = \{\mathcal{P}_1,\dots,\mathcal{P}_r\}$, where $\mathcal{P}_i$ is a polarization of c_i, $\forall i$ and for $i \neq j$ we *have $\mathcal{P}_i \cap \mathcal{P}_j = 0$ and $\{T_i : T_i \notin \mathcal{P}_j forj = 1, \dots, r\}$ is a polarization of w, where w is the boundary of the unbounded region.*

Proposition 1.5. *G has a polarization.*

Proof: see [3]

Example 1.1. *Let G be a graph on vertex set $V = \{x_1, \dots, x_4\}$*



$c = T_{i_1}, \dots T_{i_t} = T_1, T_2, T_3, T_4$. *A polarization of c is the set $\{T_1, T_3\}$ or the set $\{T_2, T_4\}$ and the set $\{T_1T_3 - T_2T_4\}$ is a Groebner basis for the toric ideal of $K[G]$.*

By a chosen polarization $\mathcal{P}$ of G, we can define a term order on the monomial of $K[G]$ such that its toric ideal has a Groebner basis consisting of atomic cycles in G [1].

Let $B_1 = K[U_1, \dots, U_r]$, and
$\Phi : B \to B_1$, $\Phi(T_k) = 0$, if $T_k \in \mathcal{P}^c$, $\Phi(T_k) = U_i$, if $T_k \in \mathcal{P}_i$,where $\mathcal{P}^c = \{T_1, \dots, T_q\} \setminus \bigcup \mathcal{P}_i$.

Theorem 1.1. *$M(G) = \{\Phi(T_c) : c$ is a cycle$\}$ is a Groebner basis for $\Phi(P(G))$ w.r.t. $>_{B_1}$.* [1]

Theorem 1.2. *$B(G) = \{\Phi(T_c) : c$ is an atomic cycle$\}$ is a Groebner basis for $\Phi(A(G))$ w.r.t. $>_{B_1}$.* [1]

2. Regularity of ideals associated to bipartite planar graphs

The definition of the Castelnuovo -Munford regularity of a graded module M on $R = k[X_1, \dots, X_n]$ is the following: If
$0 \to F_n \to \dots \to F_1 \to F_0 \to M \to 0$ is a graded minimal free resolution of M and if b_i is the maximum degree of the generators of the free module F_I then

$$reg(M) = sup\{b_i - i, i \geq 0\}.$$

In other words, $reg(M)$ is the smallest integer m such that for every j, the j-th syzygy module M is generated in degree $\leq m + j$, hence $reg(M) = sup\{\beta_{i,i+j} \geq 0\}$, $\beta_{i,k}$ are the graded Betti numbers of M.

Theorem 2.1. *(**Green**, [5]) Let $R = K[X_1, \ldots, X_n]$ be a polynomial ring. For any ideal $I \subset R$, $reg(I) \le reg(in_< I)$, where $in_< I = in_<(\overline{G})$, with $\overline{G}$ Groebner basis for I.Then $reg(I) \le reg(\overline{G})$.*

The aim is to calculate the Groebner basis $\overline{G}$ of the edge ideal $I(G)$ and the toric ideal $P(G)$, to find the regularity of $(\overline{G})$ and a bound for the regularity of $I(G)$ and $P(G)$ (for $P(G)$ via results about polarization).

1. Regularity of the edge ideal $I(G)$. We can give a conjecture for two particular classes of graphs, called St_r and $B_{2r'}$. These graphs are planar and divide the plane in r and $2r'$ regions respectively . Next, we consider the minimal free resolution of the edge ideal associated to these classes and we study the regularity of $I(G)$ linked to the number of the regions.

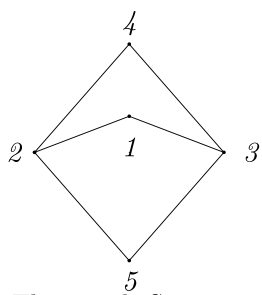
Conjecture 1): Let $G = St_r$ be the graph with vertex set $\{v_1, \ldots, v_{2r+1}\}$ and edge set $\{\{v_1, v_i\} : 2 \le i \le r+1\} \cup \{\{v_i, v_{i+r}\} : 2 \le i \le r+1\} \cup \{\{v_i, v_{i+r-1}\} : 3 \le i \le r+1\} \cup \{v_2, v_{2r+1}\}\}$, where r is the number of regions of G. Then $regI(G) \le r$

Conjecture 2): Let $G = B_{2r'}$ be the graph with vertex set $\{v_1, \ldots, v_{3r'+3}\}$ and edge set $\{\{v_1, v_{i+1}\} : 1 \le i \le 3r'+2, i \ne r'+1, 2r'+2, 3r'+3\} \cup \{\{v_i, v_{i+r'+1}\} : 1 \le i \le 2r'+2\}$, where $2r'$ is the number of regions.Then $regI(G) \le 2r'+1$.

Results:

The conjecture 1) is true for $r \in \{2, 3, 4, 5, 6, 7, 8\}$.
The conjecture 2)is true for $r' \in \{1, 2, 3, 4\}$.

Example 2.1. *Let $G = St_2$ be the bipartite complete graph on vertex set $V = \{x_1, x_2, x_3, x_4, x_5\}$*

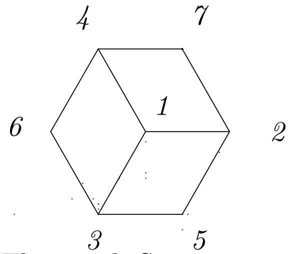


The graph St 2

$I(G) = (X_1X_2, X_1X_3, X_2X_4, X_3X_4, X_2X_5, X_3X_5)$, the minimal free resolution of $I(G)$ is

$0 \to R(-5) \to R^5(-4) \to R^9(-3) \to R^6(-2) \to R/I(G) \to 0$,
$regI(G) = 2$.

$G = St_3$,

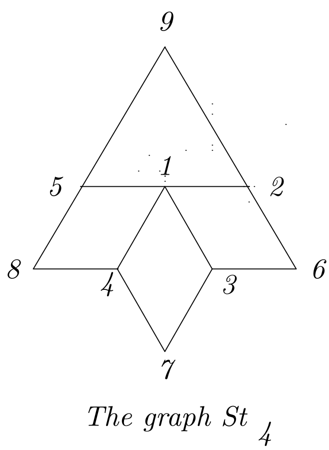


The graph St_3

$I(G) = (X_1X_2, X_1X_3, X_1X_4, X_2X_5, X_3X_6, X_4X_7, X_3X_5, X_4X_6, X_2X_7)$, *the minimal free resolution of* $I(G)$ *is*

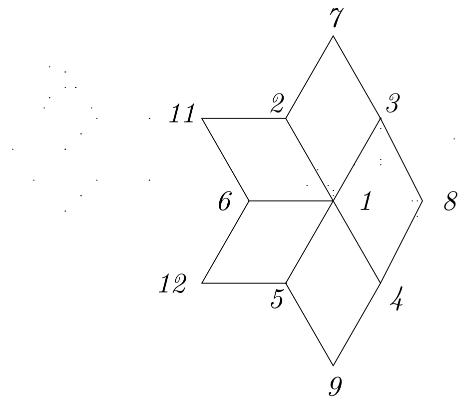
$0 \to R^2(-7) \to R^8(-6) \to R^7(-4) \oplus R^9(-5) \to R^{15}(-3) \oplus R^3(-4) \to R^9(-2) \to R/I(G) \to 0$,
$regI(G) = 3$.

$G = St_4$, $regI(G) = 4$



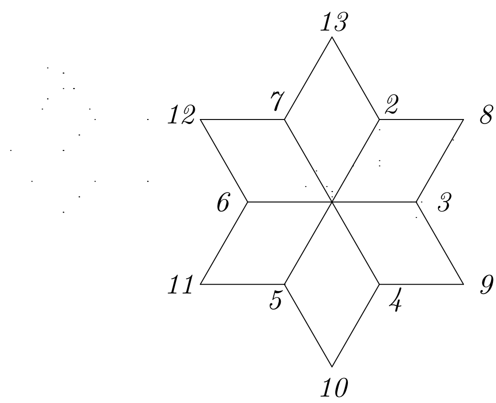
The graph St_4

$G = St_5$, $regI(G) = 4$



The graph St $_5$

$G = St_6$,



The graph St $_6$

the minimal graded resolution of $I(G)$ *is:*

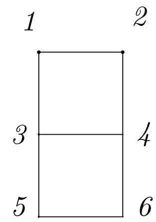
$0 \to R^2(-13) \to R^{14}(-12) \to R^{40}(-10) \oplus R^{30}(-11)$
$\to R(-7) \oplus R^{160}(-9) \oplus R^{30}(-10) \to R^6(-6) \oplus R^{42}(-7) \oplus R^{240}(-8) \oplus R^{15}(-9)$
$\to R^{15}(-5) \oplus R^{126}(-6)(+)R^{160}(-7) \oplus R^3(-8) \to R^{32}(-4) \oplus R^{126}(-5) \oplus$
$R^{40}(-6) \to R^{39}(-3) \oplus R^{42}(-4) \to R^{18}(-2)$.

$regI(G) = 5$

$G = St_7$, $regI(G) = 6$

$G = St_8$, $regI(G) = 6$

Example 2.2. *Let* $G = B_2$ *be a bipartite planar graph,*



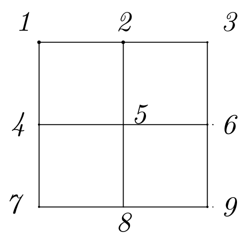
The graph B_2

$I(G) = (X_1X_2, X_3X_4, X_5X_6, X_1X_3, X_2X_4, X_4X_6, X_3X_5)$, *the minimal free resolution of* $I(G)$ *is*

$0 \to R(-6) \to R^4(-4) \oplus R^2(-5) \to R^{10}(-3) \oplus R(-4) \to R^7(-2) \to R/I(G) \to 0$,

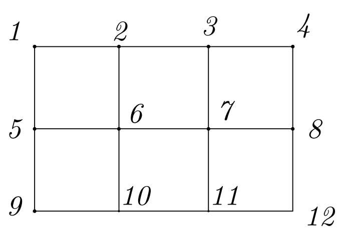
$regI(G) = 3$.

$G = B_4$, $regI(G) = 4$.



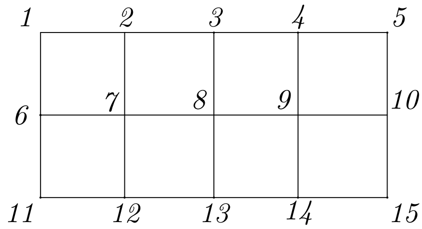
The graph B_4

For $r' = 3$, $G = B_6$, $regI(G) = 4$.



The graph B_6

For $r' = 4$, $G = B_8$, $regI(G) = 5$.



The graph B_8

2. In order to compute the regularity for the toric ideal $P(G)$ of $K[G]$, we recall the result

Theorem 2.2. *Let* $K[G]$ *be a complete intersection and a domain. Suppose that* $htP(G) = t$ *Then* $regP(G) \leq a(t-1)$.

Proof. If $K[G]$ is a complete intersection and a domain, $P(G)$ is generated by a regular sequence of t elements of degree a. The result follows by the graded minimal Koszul resolution of $P(G)$.

In this moment, we don't know if it is easy to calculate the regularity, but this is certainly possible for the graph St_r and $B_{2r'}$. This is our program.

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RENDICONTI DEL CIRCOLO MATEMATICO DI PALERMO
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GEOMETRIC PROBABILITY PROBLEMS FOR BUFFON AND LAPLACE GRID

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Abstract. In the present paper we study problems of geometric probability for grid of Buffon (strips of constant width) and Laplace (rectangles of constant sides) with bodies tests rectangles or squares.

AMS 2000 Subject Classification. Geometric probability, stochastic geometry, random sets, random convex sets and integral geometry.
AMS Classification: 60D05, 52A22.

1.

Its $\mathcal{R}_a$ a grid of equally spaced parallel straight of equidistance a (grid of Buffon), therefore its basic cell is a strip $\mathcal{C}$ with constant width. It's also Q a random square with side constant c, with $\sqrt{2}\,c \le a$ (test body).

We demonstrate the

<u>Theorem 1</u>. The probability that a random square with side constant c intersect on a straight of the grid $\mathcal{R}_a$, with $a \ge c\sqrt{2}$, a segment at least equal to s, is

$$p_{c,a;s} = \begin{cases} \dfrac{4c-2s}{\pi a} & \text{se } 0 \le s \le c \\ \dfrac{2s-4c\sin\varphi_0}{\pi a} & \text{se } c \le s \le c\sqrt{2}, \end{cases} \tag{1}$$

where $\varphi_0 \in \left[0, \pi/4\right]$ is the angle provided by the report $\cos\varphi_0 = c/s$.

<u>Demonstration</u>. It's $\varphi \in \left[0, \pi/4\right]$ the angle that one side of the square Q form with the direction of the straights of the grid $\mathcal{R}_a$ and M(φ) the center of a fixed position Q(φ) of the square Q.

Assuming that Q(φ) intersect on the straight of $\mathcal{R}_a$ closer to M(φ) a segment with length s, we indicate with $h_s(\varphi)$ the distance between M(φ) and this straight (fig. 1).

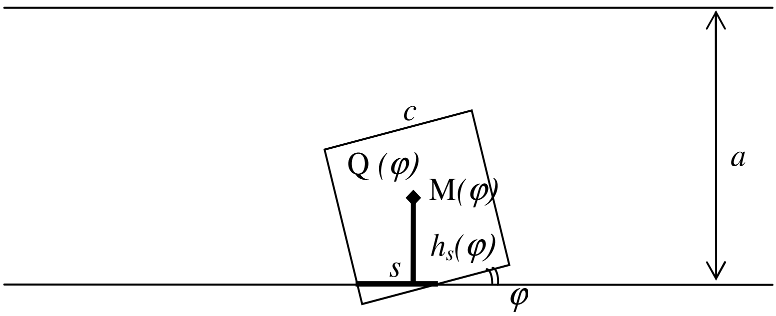

Fig. 1

It occurs easily that

$$h_s(\varphi) = \frac{c}{2}(\sin\varphi + \cos\varphi) - s\sin\varphi\cos\varphi, \tag{2}$$

while if this square does not exist, we take $h_s(\varphi)=0$.

If one side of the square Q forms the angle φ with the $\mathcal{R}_a$ straights direction, then Q intersect on a straight of $\mathcal{R}_a$ a segment with length greater than s if and only if the distance from the center M of Q from the straight is considered less than $h_s(\varphi)$, that is, if and only if M is located in one of the strips put of the figure 2.

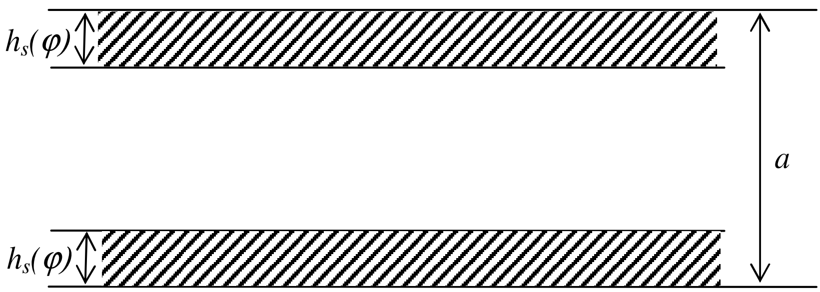

Fig. 2

If $s \le c$, we have $h_s(\varphi) = 0$, for each $\varphi \in \left[0, \pi/4\right]$.

If $c < s \le \sqrt{2}$, we have

$$h_s(\varphi) = 0 \Leftrightarrow \varphi \in [0, \varphi_o].$$

Then

$$\int_0^{\pi/4} h_s(\varphi)d\varphi = \begin{cases} \dfrac{c}{2}-\dfrac{s}{4} & \text{se } s \le c, \\ \dfrac{s}{4}-\dfrac{c}{2}\sin\varphi_0 & \text{se } c \le s \le c\sqrt{2} \end{cases}.$$

Taking into account that the probability request is

$$p_{c,a;s} = \frac{2\int_0^{\pi/4} h_s(\varphi)d\varphi}{\int_0^{\pi/4} a\, d\varphi},$$

we have the formula (1).

Consider now the same problem for the test body a rectangle T with sides c_1 and c_2 with, for example,

$$c_1 \le c_2 \le \sqrt{c_1^2 + c_2^2} \le a .$$

<u>Theorem 2</u>. The probability that a random rectangle with sides c_1 and c_2, with $c_1 \le c_2$, intersect on a straight of the grid $\mathcal{R}_a$, with $a \ge \sqrt{c_1^2 + c_2^2}$, a segment at least equal to s is

(3)

$$p_{c_1,c_2,a;s} = \begin{cases} \dfrac{2(c_1+c_2-s)}{\pi a}, & \text{se } 0 \le s \le c_1, \\ \dfrac{2c_2(1-\sin\varphi_1)}{\pi a}, & \text{se } c_1 \le s \le c_2, \\ \dfrac{2(s-c_1\sin\varphi_2 - c_2\sin\varphi_1)}{\pi a}, & \text{se } c_2 \le s \le \sqrt{c_1^2+c_2^2}, \end{cases}$$

where $\varphi_i \in \left[0, \pi/2\right]$ is the angle provided by the relation $\cos\varphi_i = \dfrac{c_i}{s},\ (i=1,2)$.

Demonstration. Its $\varphi \in \left[0, \pi/2\right]$ the angle that the small side (with length c_1) forms with the $\mathcal{R}_a$ straights direction and M(φ) the center of a fixed position T(φ) of the rectangle T (figure 3).

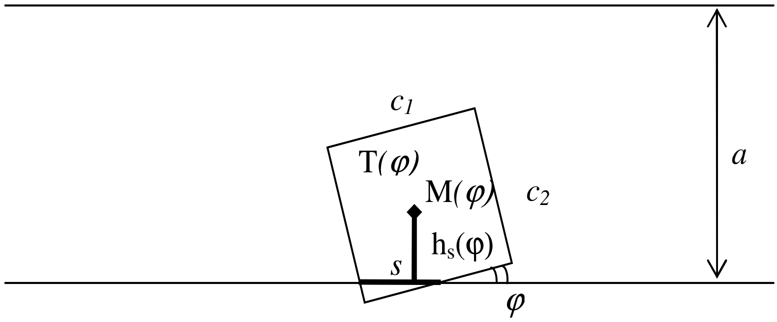

Fig. 3

Supposing that T(φ) intersects on the straight $\mathcal{R}_a$ closer to M(φ) a segment with length s. With the notation of the demonstration of theorem 1 we have

$$h_s(\varphi) = \frac{c_1}{2}\sin\varphi - \frac{c_2}{2}\cos\varphi - s\sin\varphi\cos\varphi; \tag{4}$$

while if this rectangle does not exist, we take $h_s(\varphi) = 0$.

If $s \le c_1$, we have $h_s(\varphi) > 0$ for each $\varphi \in \left[0, \pi/2\right]$ and the formula (4) gives

$$\int_0^{\pi/2} h_s(\varphi)d\varphi = \left\{ \frac{c_1 + c_2 - s}{2} \right. . \tag{5}$$

If $c_1 \le s \le c_2$, for $\varphi \in [0, \varphi_1]$ the rectangle T(φ) intersects on the straight considered of $\mathcal{R}_a$ a segment with length less than s, therefore we have $h_s(\varphi)=0$. For $\varphi \in \left[\varphi_1, \pi/2\right]$, the formula (4) gives

$$\int_0^{\pi/2} h_s(\varphi)d\varphi = \int_{\varphi_1}^{\pi/2} h_s(\varphi)d\varphi = \frac{c_2}{2}(1 - \sin\varphi) = \frac{c_2}{2}\left(1 - \sqrt{1 - \frac{c_1^2}{s^2}}\right). \tag{6}$$

If $c_2 \le s \le \sqrt{c_1^2 + c_2^2}$, we have $h_s(\varphi)=0$ for each $\varphi \in [0, \varphi_1[\cup \left[\frac{\pi}{2} - \varphi_2, \frac{\pi}{2}\right]$ while for $\varphi \in \left[\varphi_1, \pi/2 - \varphi_2\right]$, the formula (4) gives

$$(7) \qquad \int_0^{\pi/2} h_s(\varphi) d\varphi = \int_{\varphi_1}^{\frac{\pi}{2}-\varphi_2} h_s(\varphi) d\varphi = \frac{s - c_1 \sin \varphi_2 - c_2 \sin \varphi_1}{2}.$$

As in the case of the theorem 1, the probability searched is

$$p_{c_1, c_2, a; s} = \frac{2\int_0^{\pi/2} h_s(\varphi) d\varphi}{\int_0^{\pi/2} a\, d\varphi}$$

and the formulas (5), (6) and (7) give the (3).

<u>Observation</u>. If $c_1 = c_2 = c$, we have $\varphi_1 = \varphi_2 = \varphi_0$ and the formulas (1) and (3) match.

2.

It's now $\mathcal{R}_{a_1, a_2}$ the grid of straights with the cell $\mathcal{C}$ a rectangle with sides a_1 and a_2 (Laplace grid); therefore this gris is the overlap of two Buffon grids $\mathcal{R}_{a_1}$ and $\mathcal{R}_{a_2}$ with the $\mathcal{R}_{a_1}$ straights perpendicular to $\mathcal{R}_{a_2}$ straights. Ammuming as test body the same rectangle T of the theorem 2 with $\sqrt{c_1^2 + c_2^2} \le \min(a_1, a_2)$.

Let then s_1 and s_2 two not negative numbers at the same to $\min(c_1, c_2)$.

We want to calculate the probability $p_{c_1, c_2, (a_1, s_1) \vee (a_2, s_2)}$ that a test body T intersects on a $\mathcal{R}_{a_1}$ straight a segment with length at least s_1 or (without decoupling) on a $\mathcal{R}_{a_2}$ straight, a segment with length at least s_2 and the probability $p_{c_1, c_2, (a_1, s_1) \wedge (a_2, s_2)}$ that a test body intersects on a $\mathcal{R}_{a_1}$ straight a segment with length at least s_1 and on a $\mathcal{R}_{a_2}$ straight, a segment with length at least s_2

Theorem 3. We have

$$p_{c_1,c_2,(a_1,s_1)\vee(a_2,s_2)} = \frac{2}{\pi}\left(\frac{1}{a_1}+\frac{1}{a_2}\right)(c_1+c_2) - \frac{c_1^2+c_2^2}{\pi a_1 a_2} - \frac{c_1 c_2}{a_1 a_2} - \frac{2}{\pi}\left(\frac{s_1}{a_2}+\frac{s_2}{a_1}\right) - \frac{4}{3\pi}\cdot\frac{(c_1+c_2)(s_1+s_2)}{a_1 a_2}$$

and

$$(9) \qquad p_{c_1,c_2,(a_1,s_1)\wedge(a_2,s_2)} = \frac{c_1^2+c_2^2}{\pi a_1 a_2} + \frac{c_1 c_2}{a_1 a_2} + \frac{s_1 s_2}{2 a_1 a_2} - \frac{4}{3\pi}\cdot\frac{(c_1+c_2)(s_1+s_2)}{a_1 a_2}.$$

Demonstration. Indicating with $\varphi \in \left[o, \pi/2\right]$ the angle that the side with length c_1 forms with a straight of the grid $\mathcal{R}_{a_1}$, this side forms with the straights of $\mathcal{R}_{a_2}$ the angle $\frac{\pi}{2}-\varphi$. For a fixed position T(φ) of the test body that cuts a segment with length s_i on a straight of the grid $\mathcal{R}_{a_i}$, (i=1,2), closer to the center M(φ) of T(φ), its coordinates are (figure 4) (it takes account of the formula(4))

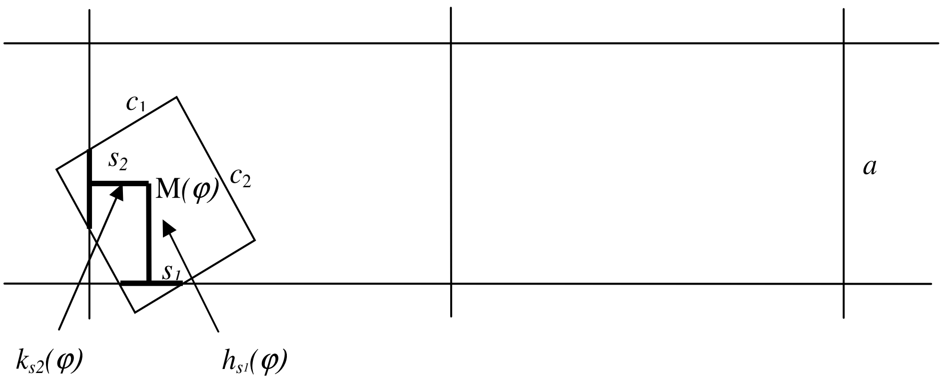

Fig. 4

$$(10) \qquad h_{s_1}(\varphi) = \frac{c_1}{2}\sin\varphi + \frac{c_2}{2}\cos\varphi - s_1\sin\varphi\cos\varphi$$

and

$$(11) \qquad k_{s_2}(\varphi) = h_{s_2}\left(\frac{\pi}{2}-\varphi\right) = \frac{c_1}{2}\cos\varphi + \frac{c_2}{2}\sin\varphi - s_2\cos\varphi\sin\varphi.$$

To calculate the request probabilities, we must to evaluate the areas $\mathcal{C}_{\vee}(\varphi)$ and $\mathcal{C}_{\wedge}(\varphi)$ of the figure 5.

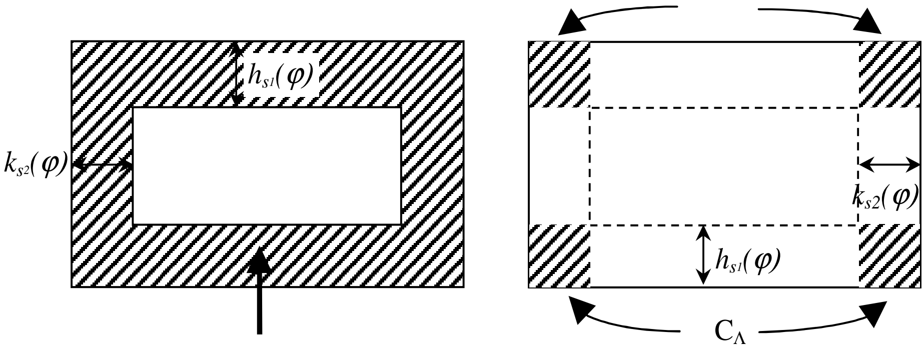

Fig. 5

We have

$$\mathcal{C}_{\vee}(\varphi) = a_1 a_2 - \left[a_1 - 2k_{s_2}(\varphi)\right]\left[a_2 - 2h_{s_1}(\varphi)\right] = (a_1 c_1 - a_2 c_2)\sin\varphi + \quad \text{area}$$
$$(a_1 c_2 + a_2 c_1)\cos\varphi - \left[2(a_1 s_1 + a_2 s_2) + c_1^2 + c_2^2\right]\sin\varphi\cos\varphi + 2(c_1 s_1 + c_2 s_2)\sin\varphi\cos$$
$$2(c_2 s_1 + c_1 s_2)\sin^2\varphi\cos\varphi - 4 s_1 s_2 \sin^2\varphi\cos^2\varphi - c_1 c_2,$$

area $\mathcal{C}_{\wedge}(\varphi) = 4k_{s_2}(\varphi)h_{s_1}(\varphi) = \left(c_1^2 + c_2^2\right)\sin\varphi\cos\varphi + c_1 c_2 + 4 s_1 s_2 \sin^2\varphi\cos^2\varphi$
$$-2(c_1 s_1 + c_2 s_2)\sin\varphi\cos^2\varphi - 2(c_2 s_1 + c_1 s_2)\sin^2\varphi\cos\varphi,$$

therefore

(12)
$$\int_0^{\pi/2} \text{area}\,\mathcal{C}_{\vee}(\varphi)d\varphi = (a_1 + a_2)(c_1 + c_2) - a_1 s_1 - a_2 s_2 - \frac{1}{2}\left(c_1^2 + c_2^2\right) - \frac{\pi}{2}c_1 c_2$$
$$-\frac{\pi}{4}s_1 s_2 + \frac{2}{3}(c_1 + c_2)(s_1 + s_2),$$

(13) $$\int_0^{\pi/2} \text{area}\,\mathcal{C}_{\wedge}(\varphi)d\varphi = \frac{1}{2}\left(c_1^2 + c_2^2\right) + \frac{\pi}{2}c_1 c_2 + \frac{\pi}{4}s_1 s_2 - \frac{2}{3}(c_1 + c_2)(s_1 + s_2).$$

Taking into account that

$$p_{c_1,c_2,(a_1,s_1)\vee(a_2,s_2)} = \frac{\int_0^{\pi/2} \text{area}\, \mathcal{C}_{\vee}(\varphi)\, d\varphi}{\int_0^{\pi/2} (\text{area}\, \mathcal{C})\, d\varphi}$$

and

$$p_{c_1,c_2,(a_1,s_1)\wedge(a_2,s_2)} = \frac{\int_0^{\pi/2} \text{area}\, \mathcal{C}_{\wedge}(\varphi)\, d\varphi}{\int_0^{\pi/2} (\text{area}\, \mathcal{C})\, d\varphi},$$

The formulas (12) and (13) give the probability (8) and (9).

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Dislocation influences on the dynamics of piezoelectric crystals

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Abstract

The aim of this paper is to derive a linearized theory to describe the behaviour of anisotropic piezoelectric crystals defective by dislocations, using an unconventional model based on the extended irreversible thermodynamics developed in previous papers, in which a dislocation tensor, its gradient and its flux are introduced as internal variables in the state vector. The linearized field equations are established.

1 Introduction

In previous papers [1] and [2], in the framework of extended irreversible thermodynamics, a unconventional model for piezoelectric crystals was developed, introducing a dislocation tensor, its gradient and its flux as internal variables, and the laws of state, the entropy production, the constitutive relations were established for isotropic and anisotropic media. Furthermore, within this model in [3] dissipative processes were studied for these materials. The objective of this paper is to achieve a linearized theory which describes the effects of dislocation lines on the behaviour of defective anisotropic piezoelectric crystals, taking into account the results obtained in [1], [2] and [3]. These materials can present metallurgical defects (for example inclusions, cavities, microfissures, dislocations) that sometimes can selfpropagate because of changed surrounding conditions that are favorable. The origin of the piezoelectricity comes from the appearance of an electric polarization in a crystal of a appropriate symmetry, when the latter is subjected to a pressure (direct effect) and the deformation of such a crystal when subjected to an electric field (inverse effect) [4]. The word "piezo" is greek and translates"push". This effect known as piezoelectricity was discovered by brothers Pierre and Jacques Curie in 1880. Piezoelectric crystals have been used in many fundamental technological sectors: in systems using surface acoustic waves for nondestructive testing, in radar technology, in the use of thin films to produce very high frequency vibrations, in the construction of intermodulators and acoustic convolves, in ultrasonics. Quartz crystals are used for watch crystals and for precise frequency reference crystals for radio transmitters. Rochelle salt produces a comparatively large voltage upon compression and are used in microphone crystal. Barium titanate, lead zirconate and lead titanate are ceramic materials which exhibit piezoelectricity and are used in ultrasonic transducers as well as microphones.

2 Continuum description of defective piezoelectric crystals, fundamental laws and constitutive equations

Now, we recall the model developed in [1] and [2] in order to study piezoelectric media, where dislocation lines form a structure which influences many physical phenomena and takes part in interactions with physical fields occurring in the body. This structure resembles a system or network of infinitesimally thin pores or capillary channels [5]. Their existence should not be omitted in the analysis of kinetic processes as diffusion of mass or charges, transport of heat, etc. The dislocation lines disturb the periodicity of the crystal lattice [6] (see Fig.1). The interatomic distances are not con-

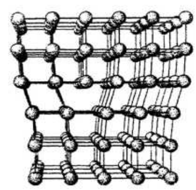

Figure 1: An edge dislocation structure (after [6]).

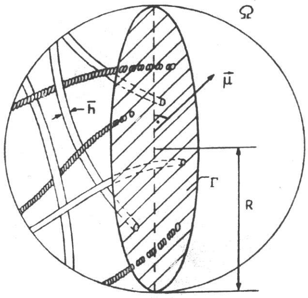

Figure 2: Characteristics of the pore-core structure ($\bar{h} \ll R$) (after [7])

served in the direct neighborhood of the dislocation line in comparison to distances in the remaining part of the lattice. The diameter of the core is comparable with the lattice parameter and its shape depends on the kind of dislocation (see Fig. 2, after [7]). In this paper we use a *dislocation core tensor* introduced in [7] by one of us (B. Maruszewski) and based on Kubik's ideas concerning a very interesting model of a porous body filled by fluid [8]. In such a medium Kubik introduces a so-called structural permeability tensor, responsible for the structure of a system of thin capillary channels, by means of the following relation

$$\bar{v}(\mathbf{x})_i = r_{ij}(\mathbf{x}, \boldsymbol{\mu}) \overset{*}{v}_j (\mathbf{x}, \boldsymbol{\mu}), \qquad \text{with} \tag{1}$$

$$\bar{\mathbf{v}}(\mathbf{x}) = \frac{1}{\Omega} \int_{\Omega^p} \mathbf{v}(\boldsymbol{\xi}) d\Omega, \;\; \boldsymbol{\xi} \in \Omega^p, \quad \text{and} \quad \overset{*}{\mathbf{v}}(\mathbf{x}, \boldsymbol{\mu}) = \frac{1}{\Gamma^p} \int_{\Gamma} \mathbf{v}(\boldsymbol{\xi}) d\Gamma, \;\; \boldsymbol{\xi} \in \Gamma^p, \tag{2}$$

where $\bar{\mathbf{v}}$ is the bulk-volume average of the fluid velocity, $\overset{*}{\mathbf{v}}$ is the corresponding pore-area average of the fluid velocity, $\Omega = \Omega^s + \Omega^p$ is a representative elementary sphere volume of a porous skeleton filled with fluid, large enough to provide a representation of all the statistical properties of the pore space Ω^p (being Ω^s the solid space), Γ is the cross section of central sphere with normal vector $\boldsymbol{\mu}$ and Γ^p is the pore area of Γ. Equation (1) gives a linear mapping between the bulk volume average fluid velocity and the local velocity of fluid particles passing through the pore area Γ^p. Now, following [7], taking into account the previous definitions, for any flux q_i of some physical field transported trough a cobweb of lines, we postulate that

$$\bar{q}(\mathbf{x})_i = r_{ij}(\mathbf{x}, \boldsymbol{\mu}) \overset{*}{q}_j (\mathbf{x}, \boldsymbol{\mu}), \quad \text{where} \quad r_{ij}(\mathbf{x}, \boldsymbol{\mu}) = \Gamma a_{ij}(\mathbf{x}, \boldsymbol{\mu}). \tag{3}$$

In equation (3) the tensor r_{ij} expresses a structure of dislocation cores and a_{ij} is a new tensor called *dislocation core tensor* that refers r_{ij} to the surface Γ. It expresses the core structure and deals with the anisotropy of the crystal (dependence on $\mathbf{x}$ and $\boldsymbol{\mu}$). Its unit is m^{-2}. Moreover, the components of a_{ij} form a kind of continuous representation of the number of dislocations which cross the surface Γ. Investigations show that a_{ij} is also dependent on time.

Now, we recall the fundamental laws and the results obtained in [1] and [2] in the framework of the extended rational thermodynamics with internal variables, where a defective piezoelectric crystal is considered in which the following fields interact with each other: the thermal field described by the temperature T and the heat flux q_i; the electromagnetic field described by the electromotive intensity $\mathcal{E}_i$ (the electric field referred to an element of matter at time t, i.e. the so called comoving frame $\mathcal{K}_c$) and the magnetic induction B_i per unit volume; the dislocation field described by the dislocation density tensor a_{ij} and the dislocation flux $\mathcal{V}_{ijk}$; the elastic field described by the total stress tensor σ_{ij} (in general non symmetric) and the small strain tensor ε_{ij} defined as $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$, where u_i are the components of the displacement vector. We use the standard Cartesian tensor notation in a rectangular coordinate system. We refer the motion of our material system to a current configuration $\mathcal{K}_t$.

Then, the independent variables are represented by the set

$$C = \{\varepsilon_{ij}, \mathcal{E}_i, B_i, T, a_{ij}, \mathcal{V}_{ijk}, q_i, T_{,i}, a_{ij,k}\}\,. \tag{4}$$

This specific choice shows that the relaxation properties of the thermal field, the dislocation field are taken into account. However, we ignore the corresponding effect for the mechanical properties so that σ_{ij} is not in the set (4) (the viscoelastic properties of the material are excluded).

All the processes occurring in the considered body are governed by three groups of laws. The first one deals with *the electromagnetic field* governed by Maxwell's equations in the following form

$$\in_{ijk} E_{k,j} + \frac{\partial B_i}{\partial t} = 0, \quad \in_{ijk} H_{k,j} - \frac{\partial D_i}{\partial t} = 0, \quad D_{i,i} = 0, \quad B_{i,i} = 0, \tag{5}$$

where there are neither currents nor electric charge, the magnetic field, the electric field and the magnetization per unit volume are given by

$$H_i = \frac{1}{\mu_0} B_i, \qquad E_i = \frac{1}{\varepsilon_0}(D_i - P_i), \qquad M_i = 0, \tag{6}$$

ε_0, μ_0 denote the permittivity and permeability of vacuum and D_i and P_i are the electric displacement and the electric polarization per unit volume.

The second group deals with the following classical balance equations:

the continuity equation $\dot{\rho} + \rho v_{i,i} = 0$,
where ρ denotes the mass density, v_i is the velocity of the body point and an superimposed dot denotes the material derivative;

the momentum balance

$$\rho \dot{v}_i - \sigma_{ji,j} - \in_{ijk} \overset{\Delta}{P}_j B_k - P_j \mathcal{E}_{i,j} - f_i = 0, \quad \text{where} \tag{7}$$

$$\overset{\Delta}{P}_i = \dot{P}_i + P_i v_{k,k} - P_k v_{i,k}, \qquad \mathcal{E}_i = E_i + \in_{ijk} v_j B_k, \tag{8}$$

and f_i is a body force;

the moment of momentum balance $\in_{ijk} \sigma_{jk} + c_i = 0$,
where c_i is a couple per unit volume;

$$\textit{the internal energy balance} \quad \rho \dot{U} - \sigma_{ji} v_{i,j} - \rho \mathcal{E}_i \dot{\mathcal{P}}_i + q_{i,i} - \rho r = 0, \tag{9}$$

where U is the internal energy density, $\mathcal{P}_i = \frac{P_i}{\rho}$ and r is the heat source distribution. Mass forces and heat source distribution will be neglected in the sequel. The third group of laws deals with *the rate properties of the dislocation core tensor, the dislocation flux and the heat flux*, respectively

$$\overset{*}{a}_{ij} + \mathcal{V}_{ijk,k} - A_{ij}(C) = 0, \quad \overset{*}{\mathcal{V}}_{ijk} - V_{ijk}(C) = 0, \quad \overset{*}{q}_i - Q_i(C) = 0, \tag{10}$$

where $\overset{*}{a}_{ij} = \dot{a}_{ij} - \Omega_{ik} a_{kj} - \Omega_{jk} a_{ik}$, $\overset{*}{\mathcal{V}}_{ijk} = \dot{\mathcal{V}}_{ijk} - \Omega_{il} \mathcal{V}_{ljk} - \Omega_{jl} \mathcal{V}_{ilk} - \Omega_{kl} \mathcal{V}_{ijl}$, $\overset{*}{q}_i = \dot{q}_i - \Omega_{ij} q_j$, Ω_{ij} is the antisymmetric part of the velocity gradient $v_{i,j}$, A_{ij} is the source-like term which may deal with the annihilation of dislocations of opposite signs, V_{ij} is the source term for the dislocation flux, Q_i is the source term for the heat flux and the superimposed asterisk denotes the Zaremba-Jaumann derivative (see [9] for the form of these equations). To be sure that the physical processes occurring in the body considered are real, all the admissible solutions of the proposed evolution equations have to satisfy the following *entropy inequality*

$$\rho \dot{S} + \nabla \cdot \mathbf{\Phi} + \tilde{\sigma} = 0, \quad \text{where} \quad \mathbf{\Phi} = \frac{1}{T}\mathbf{q} + \mathbf{k}, \quad \textit{and} \quad \tilde{\sigma} \geq 0, \tag{11}$$

with $\mathbf{\Phi}$ the entropy flux, $\mathbf{k}$ an additional term called extra entropy flux density and $\tilde{\sigma}$ the entropy production.

The set of the constitutive functions are

$$\mathbf{W} = \left\{ \tau_{ij}, P_i, c_i, U, A_{ij}, V_{ijk}, Q_i, S, \Phi_i, \pi_{ij}, \Pi^A_{ijk}, \Pi^Q_i \right\}, \tag{12}$$

being π_{ij}, Π^A_{ijk}, Π^Q_i opportune affinities. In [1] and [2] constitutive equations $\mathbf{W} = \tilde{\mathbf{W}}(C)$ were obtained for isotropic and anisotropic ferroelastic crystals, respectively. In [1] the entropy inequality (11) was analyzed by Liu's theorem [10], and the constitutive relations were obtained with the help of isotropic polynomial representations of proper constitutive functions satisfying the objectivity and material frame indifference principles (see [11], [12] and [13]). In [2] the constitutive relations were obtained expanding the free energy in Taylor's series with respect to a particular equilibrium state.

Then, in [1] several characteristic groups of expressions were deduced: *the laws of state*

$$\rho\frac{\partial F}{\partial \varepsilon_{ij}} = \sigma_{ij} + \mathcal{E}_s P_s \delta_{ij}, \quad \rho\frac{\partial F}{\partial \mathcal{E}_i} = -P_i, \quad \rho\frac{\partial F}{\partial B_i} = 0, \tag{13}$$

$$\frac{\partial F}{\partial T} = -S, \quad \frac{\partial F}{\partial T_{,i}} = 0, \quad \frac{\partial F}{\partial a_{ij,k}} = 0; \tag{14}$$

the affinities $$\rho\frac{\partial F}{\partial a_{ij}} = \pi_{ij}, \qquad \rho\frac{\partial F}{\partial \mathcal{V}_{ijk}} \equiv \Pi^A_{ijk}, \qquad \rho\frac{\partial F}{\partial q_i} \equiv \Pi^Q_i; \tag{15}$$

the relations pertaining to the *flux-like properties* of the processes

$$\frac{\partial K_k}{\partial \varepsilon_{ij}} = 0, \qquad \frac{\partial K_k}{\partial \mathcal{E}_i} = -v_k P_i, \qquad \frac{\partial K_k}{\partial B_i} = 0, \qquad \frac{\partial K_k}{\partial T_{,i}} = 0\ , \tag{16}$$

$$\frac{\partial K_k}{\partial \mathcal{V}_{ijn}} = \pi_{ij}\delta_{kn} + \Pi^A_{ijk} v_n, \qquad \frac{\partial K_k}{\partial a_{ij,n}} = 0, \qquad \frac{\partial K_k}{\partial q_i} = -\delta_{ik} + \Pi^Q_i v_k, \tag{17}$$

where the free energy density F and the flux vector K_i are defined by

$$F = U - TS - \frac{1}{\rho}\mathcal{E}_i P_i, \qquad K_i = \rho F v_i - T\Phi_i; \tag{18}$$

the residual inequality having the form

$$T\frac{\partial \Phi_k}{\partial a_{ij}} a_{ij,k} + T\frac{\partial \Phi_k}{\partial T} T_{,k} - \pi_{ij} A_{ij} - \Pi^A_{ijk} V_{ijk} - \Pi^Q_i Q_i \geq 0; \tag{19}$$

and the following relevant results:

$$K_k = -q_k + \pi_{ij}\mathcal{V}_{ijk+} + \rho v_k F, \qquad \Phi_k = \frac{1}{T}\left(q_k - \pi_{ij}\mathcal{V}_{ijk}\right), \tag{20}$$

$$F = F(\varepsilon_{ij}, \mathcal{E}_i, T, a_{ij}, \mathcal{V}_{ijk}, q_i), \tag{21}$$

the couple c_i vanishes and the stress tensor τ_{ij} is symmetric. Furthermore, three groups of constitutive functions were deduced:

$$\sigma_{ij} = \sigma_{ij}(C_1), \quad P_i = P_i(C_1), \quad S = S(C_1), \quad \pi_{ij} = \pi_{ij}(C_1), \tag{22}$$

where $\quad C_1 = \{\varepsilon_{ij}, \mathcal{E}_i, T, a_{ij}\}$,

$$\Pi_i^Q = \Pi_i^Q(C_2), \qquad \Pi_{ijk}^A = \Pi_i^A(C_2), \quad \text{where} \quad C_2 = \{\mathcal{V}_{ijk}, q_i\} \tag{23}$$

and $\qquad A_{ij} = A_{ij}(C), \qquad V_{ijk} = V_{ijk}(C), \qquad Q_k = Q_k(C).$

In [2] the constitutive theory for anisotropic media was formulated, expanding the free energy F into Taylor's series with respect to an equilibrium state and confining the consideration to the quadratic terms. Considering very small deviations with respect to equilibrium (with the subscript "$_0$" referring to the equilibrium state), it was assumed

$$T = T_0 + \theta, \qquad \left|\frac{\theta}{T_0}\right| \ll 1, \qquad \text{and} \quad M = M_0 + m, \quad \left|\frac{m}{M_0}\right| \ll 1, \tag{24}$$

where T_0 denotes the room temperature, M stands for all considered interacting fields $M = \{\varepsilon_{ij}, \mathcal{E}_i, B_i, T, a_{ij}, \mathcal{V}_{ijk}, q_i, T_{,i}, a_{ij,k}\}$, θ and m are very small deviations and M_0 defines a natural state of the body in which

$$M_0 = 0. \tag{25}$$

For the dislocation field the small deviation m was called α_{ij}.

Finally, in [2], by virtue of the laws of state, the approximated expression for the free energy and the rate equations for the fluxes, the following expressions were derived

$$P_i = \rho p_i = h_{ijl}\varepsilon_{jl} + \chi_{il}\mathcal{E}_l + \rho\lambda_i^\theta\theta - \alpha_{ijl}^{a\mathcal{E}}\alpha_{jl}, \tag{26}$$

$$\sigma_{ij} = c_{ijlm}\varepsilon_{lm} - h_{ijl}\mathcal{E}_l - \lambda_{ij}^\theta\theta + \alpha_{ijlm}^{a\epsilon}\alpha_{lm} - \mathcal{E}_s P_s \delta_{ij}, \tag{27}$$

$$S = \frac{\lambda_{ij}^\theta}{\rho}\varepsilon_{ij} + \lambda_i^\theta\mathcal{E}_i + \frac{c}{T_0}\theta - \frac{\alpha_{ij}^{a\theta}}{\rho}\alpha_{ij}, \tag{28}$$

$$\pi_{ij} = \alpha_{ijlm}^{a\epsilon}\varepsilon_{lm} + \alpha_{ijl}^{a\mathcal{E}}\mathcal{E}_l + \alpha_{ij}^{a\theta}\theta + \alpha_{ijlm}^{aa}\alpha_{lm}, \tag{29}$$

$$\Pi_{ijk}^A = \alpha_{ijklmn}^{\nu\nu}\mathcal{V}_{lmn} + \alpha_{ijkl}^{\nu q}q_l, \qquad \Pi_i^Q = \alpha_{ijkl}^{\nu q}\mathcal{V}_{jkl} + \alpha_{ij}^{qq}q_j, \tag{30}$$

$$\overset{*}{q}_i = \delta_{ij}^1\mathcal{E}_j + \delta_{ij}^2\theta_{,j} + \delta_{ij}^3 q_j + \delta_{ijk}^4\alpha_{jk} + \delta_{ijk}^5\varepsilon_{jk} + \delta_{ijkl}^6\mathcal{V}_{jkl} + \delta_{ijkl}^7\alpha_{jk,l}, \tag{31}$$

$$\begin{aligned}\overset{*}{\alpha}_{ij} = {} & \beta_{ijk}^1\mathcal{E}_k + \beta_{ijk}^2\theta_{,k} + \beta_{ijk}^3 q_k + \beta_{ijkl}^4\varepsilon_{kl} + \\ & + \beta_{ijkl}^5\alpha_{kl} + \beta_{ijklm}^6\mathcal{V}_{klm} + \beta_{ijklm}^7\alpha_{kl,m},\end{aligned} \tag{32}$$

$$\begin{aligned}\overset{*}{\mathcal{V}}_{ijk} = {} & \gamma_{ijkl}^1\mathcal{E}_l + \gamma_{ijkl}^2\theta_{,l} + \gamma_{ijkl}^3 q_l + \gamma_{ijklm}^4\varepsilon_{lm} + \gamma_{ijklm}^5\alpha_{lm} + \\ & + \gamma_{ijklmn}^6\mathcal{V}_{lmn} + \gamma_{ijklmn}^7\alpha_{lm,n}.\end{aligned} \tag{33}$$

In equs. (26)-(33) the constant phenomenological coefficients introduced satisfy the following symmetric relations

$$c_{ijlm} = c_{lmij} = c_{jilm} = c_{ijml} = c_{jiml} = c_{mlij} = c_{mlji} = c_{lmji}, \tag{34}$$

$$\lambda^{\theta}_{ij} = \lambda^{\theta}_{ji}, \quad h_{ijl} = h_{lij} = h_{jil} = h_{lji}, \quad \alpha^{a\mathcal{E}}_{ijl} = \alpha^{a\mathcal{E}}_{lij}, \tag{35}$$

$$\alpha^{a\epsilon}_{ijlm} = \alpha^{a\epsilon}_{lmji} = \alpha^{a\epsilon}_{lmij} = \alpha^{a\epsilon}_{jilm}, \quad \chi_{il} = \chi_{li}, \quad \alpha^{aa}_{ijlm} = \alpha^{aa}_{lmij}, \tag{36}$$

$$\alpha^{\nu\nu}_{ijklmn} = \alpha^{\nu\nu}_{lmnijk}, \quad \alpha^{\nu q}_{ijkl} = \alpha^{\nu q}_{lijk}, \qquad \alpha^{qq}_{ij} = \alpha^{qq}_{ji}, \tag{37}$$

and the following notations are introduced: c denotes the specific heat, c_{ijlm} is the elastic tensor, λ^{θ}_{ij} are the thermoelastic constants, χ_{il} is the electrical susceptibility tensor, h_{ijl} are the piezoelectric constants and other quantities express interactions among the various effects involved in the system. In equ. (32) we have taken into consideration $\mathcal{V}_{ijk,k} = 0$. Furthermore, the rate equation for the heat flux (31) generalizes Vernotte - Cattaneo relation and, when it is possible to identify the Zaremba - Jaumann derivative with the material derivative, denoting by τ_{ij} a relaxation time tensor associated to the heat flux, it takes the form

$$\tau_{ij}\dot{q}_j = -q_i - \chi^1_{ij}\theta_{,j} - \chi^2_{ij}\mathcal{E}_j + \chi^3_{ijk}\alpha_{jk} + \chi^4_{ijk}\varepsilon_{jk} + \chi^5_{ijkl}\mathcal{V}_{jkl} + \chi^6_{ijkl}\alpha_{jk,l}. \tag{38}$$

In the case that $\tau_{ij} = \tau\delta_{ij}$ equ. (38) becomes

$$\tau\dot{q}_i = -q_i - \chi^1_{ij}\theta_{,j} - \chi^2_{ij}\mathcal{E}_j + \chi^3_{ijk}\alpha_{jk} + \chi^4_{ijk}\varepsilon_{jk} + \chi^5_{ijkl}\mathcal{V}_{jkl} + \chi^6_{ijkl}\alpha_{jk,l}. \tag{39}$$

3 Field equations

Now, we consider the case of defective anisotropic piezoelectrics and, to obtain field equations which allow to consider and solve analytically and/or numerically particular problems, we linearize the developed theory, obtaining a mathematical model to describe the physical reality in many situations. Introducing the Legendre transformation $(18)_1$, considering the material derivative of the free energy F (see also [3])

$$\rho T\dot{S} = \rho\dot{U} - \rho S\dot{T} - \rho\dot{F} - \dot{\mathcal{E}}_i P_i - \mathcal{E}_i \dot{P}_i, \tag{40}$$

and taking into consideration the balance energy equation

$$\rho\dot{U} = \sigma_{ji}\dot{\varepsilon}_{ij} - q_{i,i} + \mathcal{E}_i\dot{P}_i, \tag{41}$$

(where the expression for the velocity gradient $v_{i,j} = \dot{\varepsilon}_{ji} + \Omega_{ji}$ has been used and ρ is considered practically constant), we obtain

$$\rho T\dot{S} = \sigma_{ji}\dot{\varepsilon}_{ij} - q_{i,i} - \rho\dot{T}S - \rho\dot{F} - \dot{\mathcal{E}}_i P_i \tag{42}$$

From (42), calculating the material derivative of the free energy $F = F\left(\varepsilon_{ij}, \mathcal{E}_i, \theta, \alpha_{ij}\mathcal{V}_{ijk}, q_i\right)$, we have

$$\rho T\dot{S} = \sigma_{ij}\dot{\varepsilon}_{ij} - q_{i,i} - \rho\dot{T}S - \rho\frac{\partial F}{\partial\varepsilon_{ij}}\dot{\varepsilon}_{ij} - \rho\frac{\partial F}{\partial\mathcal{E}_{\;i}}\dot{\mathcal{E}}_i +$$

$$-\rho\frac{\partial F}{\partial\theta}\dot{\theta} - \rho\frac{\partial F}{\partial\alpha_{ij}}\dot{\alpha}_{ij} - \rho\frac{\partial F}{\partial\mathcal{V}_{ijk}}\dot{\mathcal{V}}_{ijk} - \rho\frac{\partial F}{\partial q_i}\dot{q}_i - \dot{\mathcal{E}}_i P_i. \tag{43}$$

Finally, using the laws of state, the definitions of the affinities and the rate equations for the heat and dislocation fluxes and for the core dislocation tensor (in the case when it is possible to identify the Zaremba-Jaumann derivative with the material derivative), we have

$$\rho T\dot{S} = -q_{i,i} - \pi_{ij}\dot{\alpha}_{ij} - \Pi^{A}_{ijk}\dot{\mathcal{V}}_{ijk} - \Pi^{Q}_{i}\dot{q}_i - E_s P_s\dot{\varepsilon}_{ij}\delta_{ij}. \tag{44}$$

Now, we consider equations (5) for the electric magnetic fields. Calculating the partial time derivative of equ. $(5)_2$, applying the curl operator to equ. $(5)_1$ and using equ. $(6)_1$ we obtain (in the approximation $\frac{d}{dt} \cong \frac{\partial}{\partial t}$)

$$\nabla(\nabla\cdot\mathbf{E}) - \nabla^2\mathbf{E} = -\mu_0\ddot{\mathbf{D}}. \tag{45}$$

Finally, utilizing equ. $(6)_2$, we derive

$$\nabla(\nabla\cdot\mathbf{P}) + \varepsilon_0\nabla^2\mathbf{E} = \mu_0(\varepsilon_0)^2\ddot{\mathbf{E}} + \mu_0\varepsilon_0\ddot{\mathbf{P}}, \tag{46}$$

or in components

$$P_{i,ij} + \varepsilon_0 E_{i,jj} = \mu_0(\varepsilon_0)^2\ddot{E}_i + \mu_0\varepsilon_0\ddot{P}_i. \tag{47}$$

Then, we linearize the momentum balance equ.(7), the heat equation (44) and the electromagnetic field equation (47). By virtue of (24) - (25), and using (26) - (28) and (39) we obtain

the momentum balance

$$\rho_0\ddot{u}_i - c_{ijlm}\varepsilon_{lm,j} - h_{ijl}\mathcal{E}_{l,j} - \lambda^{\theta}_{ij}\theta_{,j} + \alpha^{a\epsilon}_{ijlm}\alpha_{lm,j} - \left[\mathcal{E}_{0j}P_{j,i} + \mathcal{E}_{j,i}P_{0j}\right] +$$

$$-\in_{ijk}\left[\dot{P}_j + P_{0j}\dot{u}_{s,s} - P_{0s}\dot{u}_{j,s}\right]B_{0k} - P_{0j}\left[E_{i,j} + \in_{isk}\dot{u}_{s,j}B_{0k}\right] = 0, \tag{48}$$

$$\rho\ddot{u}_i = c_{ijlm}\varepsilon_{lm,j} + h_{ijl}\mathcal{E}_{l,j} + \lambda^{\theta}_{ij}\theta_{,j} - \alpha^{a\epsilon}_{ijlm}\alpha_{lm,j}; \tag{49}$$

the heat equation

$$\rho T_0\dot{S} = -q_{i,i}, \qquad \tau\rho T_0\ddot{S} = -\tau\dot{q}_{i,i}, \tag{50}$$

$$\tau\rho T_0(\frac{\lambda^\theta_{ij}}{\rho_0}\ddot{\varepsilon}_{ij}+\lambda^\theta_i\ddot{\mathcal{E}}_i+\frac{c}{T_0}\ddot{\theta}-\frac{\alpha^{a\theta}_{ij}}{\rho_0}\ddot{\alpha}_{ij})=-\rho T_0(\frac{\lambda^\theta_{ij}}{\rho_0}\dot{\varepsilon}_{ij}+\lambda^\theta_i\dot{\mathcal{E}}_i+\frac{c}{T_0}\dot{\theta}-\frac{\alpha^{a\theta}_{ij}}{\rho_0}\dot{\alpha}_{ij})+$$

$$+\chi^1_{ij}\theta_{,ji}+\chi^2_{ij}\mathcal{E}_{j,i}-\chi^3_{ijk}\alpha_{jk,i}-\chi^4_{ijk}\varepsilon_{jk,i}-\chi^5_{ijkl}\mathcal{V}_{jkl,i}-\chi^6_{ijkl}\alpha_{jk,li}\ ; \quad (51)$$

the electromagnetic field equation

$$(h_{isl}\varepsilon_{sl,ij}+\chi_{il}\mathcal{E}_{l,ij}+\rho_0\lambda^\theta_i\theta_{,ij}-\alpha^{a\mathcal{E}}_{isl}\alpha_{sl,ij})+\varepsilon_0 E_{i,jj}=\mu_0(\varepsilon_0)^2\ddot{E}_i+$$

$$+\mu_0\varepsilon_0(h_{isl}\ddot{\varepsilon}_{sl}+\chi_{il}\ddot{\mathcal{E}}_l+\rho_0\lambda^\theta_i\ddot{\theta}-\alpha^{a\mathcal{E}}_{isl}\ddot{\alpha}_{sl}). \quad (52)$$

Equations (49) - (52) allow to describe the behaviour of anisotropic piezoelectrics defective by dislocation lines in the linear approximation.

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A BUFFON TYPE PROBLEM FOR A LATTICE OF PARALLELOGRAMS WITH OBSTACLES

Michael Retter

Abstract

I consider as test body a segment s of length l and determine the probability p_s that s, uniformly distributed in a bounded region of the Euclidean plane, intersects obstacles of a lattice R of parallelograms with sides A and B and acute angle $\alpha \in]0, \frac{\pi}{2}]$. The obstacles have the form of parallelograms with same barycenters and angle than the lattice cells.

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1 Introduction

Let be given in the Euclidean plane a lattice R whose fundamental domain $\mathcal{C}$ is parallelogram of sides A and B and acute angle $\alpha \in]0, \frac{\pi}{2}]$. Each lattice cell includes an obstacle in form of a parallelogram with same center of gravity (see Figure 1), same angle and parallel sides as the parallelogram lattice cell (see figure). Let be the sides of the obstacle of length $A-2x$, $B-2x$, where $0 < x < \min A/2, B/2$. We determine the probability and the "degree of disturbance" that a segment of length l hits the obstacle or the lattice. The distance between the obstacle and the lattice is denoted by h. The complement of the obstacle within the fundamental domain is called **inner roadway**. The respective inner roadways of the neighborhood of the fundamental domain is called the **outer roadway**(see figure 1). We denote by M the

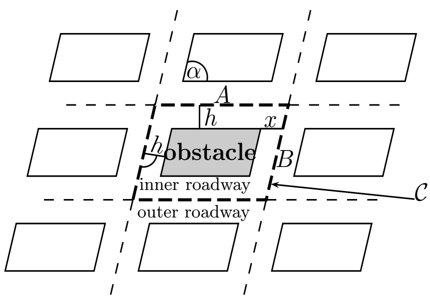

Figure 1: The fundamental domain $\mathcal{C}$

barycenter of s and d the line through M, intrinsically related to the segment. Let be φ the angle between d and the x-axis. For $\varphi \in [0, \pi[$ we obtain, except translations, all possible positions of s in respect of the lattice. The couple (M, φ) completely characterizes s and we denote (M, φ) the data of s. In the first section we want to calculate the probability that the segment is entirely placed on the inner roadway, neither intersecting the obstacle nor the border of the parallelogram lattice cells. To calculate the probability let

$\mathcal{M} :=$ set of segments of length l with barycenter in $\mathcal{C}$,

$\mathcal{N} :=$ set of all segments of length l lying entirely on the inner roadway.

Due to a formula of Stoka (see [4]) we have

$$p = \frac{\mu(\mathcal{N})}{\mu(\mathcal{M})} = \frac{\mu(\mathcal{N})}{\pi AB \sin\alpha}, \tag{1}$$

with μ Lebesgue measure. $\mu(\mathcal{N}), \mu(\mathcal{M})$ can be calculated via the **kinematic measure** in $\mathbb{E}_2$. To determine $\mu(\mathcal{N})$ we have to consider different cases, depending on the length l of the segment s.

2 The segment lies entirely on the inner roadway

We denote by $\mathcal{C}_i^j(\varphi)$ the **admissible domain** (contained in $\mathcal{C}$), with the following property: a point P is part of $\mathcal{C}_i^j(\varphi)$ if $P \in \mathcal{C}$ is barycenter of a segment s of length l, that lies entirely on the inner roadway. The area of these „admissible domains" is used to calculate $\mu(\mathcal{N})$. The index i signalizes the several cases of length l. The index j signalizes each case or each sub-interval of $[0, \pi[$ for which the admissible domain changes.

The case that s is smaller than h

We consider two cases: $\varphi \in [0, \alpha[$ and $\varphi \in [\alpha, \pi[$.
If $\varphi \in [0, \alpha]$, $\mathcal{C}_0^1(\varphi)$ (part of $\mathcal{C}$) consists of all barycenters P of segments of length l, that have

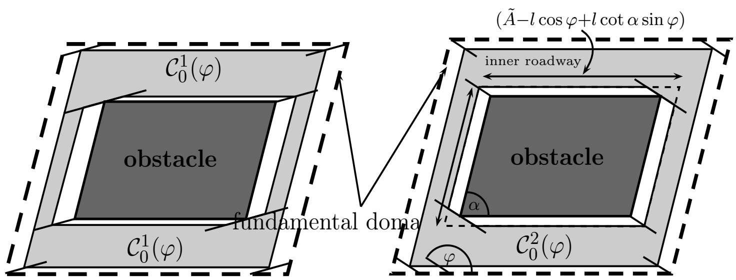

Figure 2: $\mathcal{C}_0^1(\varphi)$ and $\mathcal{C}_0^2(\varphi)$

angle φ to the x-axis and neither intersects the sides of the lattice of the outer parallelogram (i.e. the border of $\mathcal{C}$) nor the obstacle-parallelogram with sides $\tilde{A} = A - 2x$, $\tilde{B} = B - 2x$ (figure 2).
Let be $\mathcal{Z}_1(\varphi)$ the set of barycenters of segments of length l contained in $\mathcal{C}$, with angle φ to the x-axis and not hitting the border of $\mathcal{C}$. We deduce

$$area(\mathcal{Z}_1(\varphi)) = (A - l\cos\varphi + l\cot\alpha\sin\varphi) \cdot (B\sin\alpha - l\sin\varphi).$$

We define $\mathcal{Z}_3(\varphi)$ the set of barycenters of segments of length l who are hitting the obstacle-parallelogram. We observe that $\mathcal{Z}_3(\varphi)$ consists of the obstacle together with a „frame" of barycenters of segments hitting the obstacle points and being part of the inner roadway. We get:

$$area(\mathcal{Z}_3(\varphi)) = \tilde{A}\tilde{B}\sin\alpha + \tilde{A}l\sin\varphi + \tilde{B}l\sin\alpha\cos\varphi - \tilde{B}l\cos\alpha\sin\varphi.$$

Hence

$$\begin{aligned} area(\mathcal{C}_0^1(\varphi)) = \quad & area(\mathcal{Z}_1(\varphi)) - area(\mathcal{Z}_3(\varphi))2l(B-x)\sin(\varphi-\alpha) \\ & -2l(A-x)\sin\varphi + l^2\sin\varphi\tfrac{\sin(\alpha-\varphi)}{\sin\alpha} + 2x(A+B-2x)\sin\alpha. \end{aligned}$$

If $\alpha \le \varphi \le \pi$, $\mathcal{C}_0^2(\varphi)$ is defined analogously. Let $\mathcal{Z}_4(\varphi)$ denote the parallelogram with area

$$area(\mathcal{Z}_4(\varphi)) = (A + l\cos\varphi - l\cot\alpha\sin\varphi)\cdot(B\sin\alpha - l\sin\varphi).$$

Furthermore let $\mathcal{Z}_5(\varphi)$ be the parallelogram with sides of length $(\tilde{A} - l\cos\varphi + l\cot\alpha\sin\varphi)$ and $(\tilde{B}\sin\alpha + l\sin\varphi)$ minus two triangles in the corner (dotted in figure 2). The triangle's area is added up to $(-l\cos\varphi + l\cot\alpha\sin\varphi)l\sin\varphi$. Hence

$$area(\mathcal{Z}_5(\varphi)) = \tilde{A}\tilde{B}\sin\alpha + \tilde{A}l\sin\varphi - \tilde{B}l\cos\varphi\sin\alpha + \tilde{B}l\cos\alpha\sin\varphi.$$

Thus

$$\begin{aligned} & area(\mathcal{C}_0^2(\varphi)) = area(\mathcal{Z}_4(\varphi)) - area(\mathcal{Z}_5(\varphi)) = 2l(B-x)\sin(\alpha-\varphi) \\ & -2l(A-x)\sin\varphi + l^2\sin\varphi\tfrac{\sin(\varphi-\alpha)}{\sin\alpha} + 2x(A+B-2x)\sin\alpha. \end{aligned}$$

Therefore

$$\begin{aligned} & \mu(\mathcal{N}) = \int_0^\alpha d\varphi \int_{P\in\mathcal{C}_0^1(\varphi)} dP + \int_\alpha^\pi d\varphi \int_{P\in\mathcal{C}_0^2(\varphi)} dP = \int_0^\alpha area(\mathcal{C}_0^1(\varphi))d\varphi \\ & + \int_\alpha^\pi area(\mathcal{C}_0^2(\varphi))d\varphi = 2\pi x(A+B-2x)\sin\alpha - 4l(A+B-2x) + l^2\left[1 + (\tfrac{\pi}{2}-\alpha)\cot\alpha\right]. \end{aligned}$$

We immediately get

Proposition 1 *If $0 \le l \le h$, the probability p, that a small segment of length l, uniformly distributed in a bounded region of the plane, **lies entirely on the inner roadway** of the lattice, is*

$$p_s(A,B,x,\alpha,l) \quad = \quad \frac{2\pi h(A+B-2x) - 4l(A+B-2x) + l^2\left[1 + (\frac{\pi}{2}-\alpha)\cot\alpha\right]}{\pi AB\sin\alpha}.$$

The case that s is between h and x

Let be $d_1 := 2x\sin\frac{\alpha}{2}$ the "little" diagonal of the rhombus with sides x and angle α in the corners of the fundamental domain. We distinct the cases $l \le d_1$ and $l \ge d_1$.

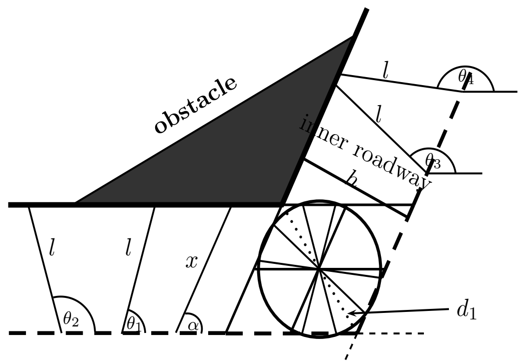

Figure 3: $l \leq d_1$

$l \leq d_1$

We define the following angles: $\theta_1 := \arcsin \frac{h}{l}$, $\theta_2 := \pi - \arcsin \frac{h}{l}$, $\theta_3 := \alpha + \arcsin \frac{h}{l}$ and $\theta_4 := \pi + \alpha - \arcsin \frac{h}{l}$.
If $l \leq d_1$, we get

$$0 \leq \alpha \leq \theta_1 \leq \theta_2 \leq \theta_3 \leq \theta_4 \leq \pi.$$

- $0 \leq \varphi \leq \alpha$

  Here we have a similar situation like the above mentioned case. The area of the admissible domain $\mathcal{C}_1^1(\varphi)$ can be calculated by the same formula as the area of the admissible domain $\mathcal{C}_0^1(\varphi)$ (see figure 2); $area(\mathcal{C}_1^1(\varphi)) = area(\mathcal{C}_0^1(\varphi))$. Hence

$$\begin{aligned}\int_0^\alpha area(\mathcal{C}_1^1(\varphi))d\varphi &= 2l\,[A+B-2x]\cos\alpha - 2l\,[A+B-2x] \\ &+2\alpha x(A+B-2x)\sin\alpha + \tfrac{1}{2}l^2\,[1-\alpha\cot\alpha]\,.\end{aligned}$$

- $\alpha \leq \varphi \leq \theta_1$

  Analogously, we get $area(\mathcal{C}_1^2(\varphi)) = area(\mathcal{C}_0^2(\varphi))$ („=" means same algebraic expression). Therefore we get

$$\begin{aligned}\int_\alpha^{\theta_1} area(\mathcal{C}_1^2(\varphi))d\varphi &= 2x(\theta_1-\alpha)(A+B-2x)\sin\alpha + 2(B-x)h\sin\alpha \\ &-\tfrac{1}{2}h^2 + 2[(B-x)\cos\alpha + (A-x)]\sqrt{l^2-h^2} - \tfrac{1}{2}h\sqrt{l^2-h^2}\cot\alpha \\ &-2l[(B-x)+(A-x)\cos\alpha] + l^2[\tfrac{1}{2}+\tfrac{1}{2}(\theta_1-\alpha)\cot\alpha].\end{aligned}$$

- $\theta_1 \leq \varphi \leq \theta_2$

  With $\theta_2 = \pi - \arcsin \frac{h}{l}$ we get a new admissible domain $\mathcal{C}_1^3(\varphi)$ with formula

$$area(\mathcal{C}_1^3(\varphi)) = 2Bh - 2lB\sin(\varphi-\alpha) - 2lx\sin\varphi + 2l^2\sin\varphi\frac{\sin(\varphi-\alpha)}{\sin\alpha} + hx\frac{\sin(\varphi-\alpha)}{\sin\varphi}$$

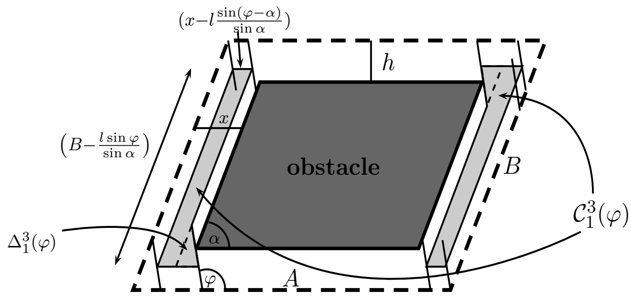

Figure 4: $\mathcal{C}_1^3(\varphi)$ for $\theta_1 \leq \varphi \leq \theta_2$

by calculating the area of 2 parallelograms with sides $(x - l\frac{\sin(\varphi-\alpha)}{\sin\alpha})$ and $(B - \frac{l\sin\varphi}{\sin\alpha})$ as well as the area of two triangles $\Delta_1^3(\varphi)$ with length x and height $h\frac{\sin(\varphi-\alpha)}{\sin\varphi}$. The triangle has area $area(\Delta_1^3(\varphi)) = \frac{1}{2}hx\frac{\sin(\varphi-\alpha)}{\sin\varphi}$. Figure 4 shows the admissible domain $\mathcal{C}_1^3(\varphi)$.
Hence

$$\begin{aligned}\int_{\theta_1}^{\theta_2} areaC_1^3(\varphi)d\varphi &= (2Bx\sin\alpha + hx\cos\alpha)(\theta_2-\theta_1) + l^2(\theta_2-\theta_1)\cot\alpha \\ &\quad +2h\sqrt{l^2-h^2}\cot\alpha - 4\sqrt{l^2-h^2}(x+B\cos\alpha).\end{aligned}$$

- $\theta_2 \leq \varphi \leq \theta_3$
  $area(\mathcal{C}_1^4(\varphi))$ can by derived by the same approach as $area(\mathcal{C}_0^2(\varphi))$.
- $\theta_3 \leq \varphi \leq \theta_4$
  Figure 5 shows the admissible domain $\mathcal{C}_1^5(\varphi)$.

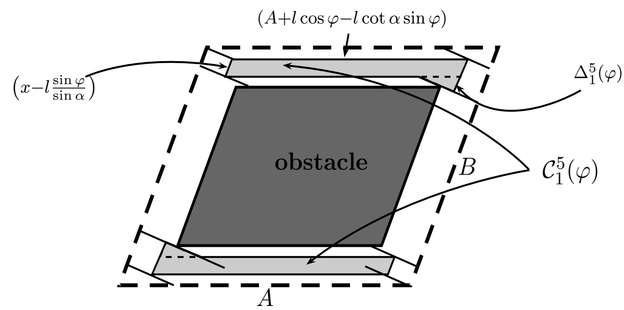

Figure 5: $\mathcal{C}_1^5(\varphi)$ for $\theta_3 \leq \varphi \leq \theta_4$

We have $area(\Delta_1^5(\varphi)) = \frac{1}{2}hx\frac{\sin\varphi}{\sin(\varphi-\alpha)}$ and therefore

$$area(\mathcal{C}_1^5(\varphi)) = 2\sin\alpha\left(x - l\frac{\sin\varphi}{\sin\alpha}\right)(A + l\cos\varphi - l\cot\alpha\sin\varphi) + 2\,area(\Delta_1^5(\varphi))$$

$$= \quad 2\,Ax\sin\alpha + 2\,lx\sin(\alpha-\varphi) - 2\,Al\sin\varphi + 2\,l^2\sin\varphi\frac{\sin(\varphi-\alpha)}{\sin\alpha} + hx\frac{\sin\varphi}{\sin(\varphi-\alpha)}$$

and $\int_{\theta_3}^{\theta_4} area(\mathcal{C}_1^5(\varphi))d\varphi = [2Ax\sin\alpha + hx\cos\alpha + l^2\cot\alpha](\theta_4-\theta_3) + 2\,(x\cos\alpha - 2A\cos\alpha - 2x)\sqrt{l^2-h^2}$.

- $\theta_4 \le \varphi \le \pi$ We deduce again

  $area(\mathcal{C}_1^6(\varphi)) = 2l(B-x)\sin(\alpha-\varphi) - 2l(A-x)\sin\varphi + l^2\sin\varphi\frac{\sin(\varphi-\alpha)}{\sin\alpha} + 2x(A+B-2x)\sin\alpha.$

This leads to the following proposition.

Proposition 2 *If $h \le l \le x$ and $l \le d_1 = 2x\sin\frac{\alpha}{2}$, the probability p that a segment of constant length l, uniformly distributed in a bounded region of the plane, lies **entirely on the inner roadway**, is*

$$\begin{aligned} p_s &= \frac{1}{\pi AB\sin\alpha}\Bigg(l^2\left(1+\left(\tfrac{3}{2}\pi-\alpha-2\arcsin\tfrac{h}{l}\right)\cot\alpha\right) - 4\,l\,(A+B-2\,x) + 2\,\pi hx\,(2+\cos\alpha) \\ &\quad +2\sqrt{l^2-h^2}\Big(2\,A+2\,B-8\,x-3x\cos\alpha\Big) + 4\,h\Big(A+B-4\,x-x\cos\alpha\Big)\arcsin\frac{h}{l}\Bigg). \end{aligned}$$

This problem is solved in [3] for needles of length $d_1 \le l \le x$, $x \le l \le d_2$, where $d_2 = 2x\cos\frac{\alpha}{2}$ the „large" diagonal of the rhombus. Furthermore this problem is solved for „large" needles, i.e. $d_2 \le l \le 2h$.

3 The segment blocks up the inner roadway

We want to regard another problem that can be solved for this lattice with two roadways: the probability that a segment blocks up the inner roadway. It is obvious that the needle must have at least length h. This problem has been solved for the cases $l \le x$, $x \le l \le d_2$ and $d_2 \le l$ in my PhD-thesis ([3]). The case $h \le l \le x$ is presented here for $l \le d_1$ and $l \ge d_1$.

Let be $\mathbf{h \le l \le x}$ and $\mathbf{l \le d_1}$

Lemma 1 *Let be $l \le d_1$. With the angles $\theta_1 = \arcsin\frac{h}{l}$, $\theta_2 = \pi - \arcsin\frac{h}{l}$, $\theta_3 = \alpha + \arcsin\frac{h}{l}$ and $\theta_4 = \pi + \alpha - \arcsin\frac{h}{l}$ the following inequalities*

$$0 \le \theta_1 \le \theta_2 \le \theta_3 \le \theta_4 \le \pi$$

hold.

Figure 6 shows the geometrically interpretation of this inequality:

$$(\pi - \theta_1) = \theta_2 \le \theta_3 = (\alpha + \theta_1) \Longleftrightarrow \frac{\pi}{2} - \frac{\alpha}{2} < \theta_1.$$

The hatched parts of the circle signalize the angles of the segment, for which the inner roadway can be blocked up. Lets consider the different cases:

- $0 \le \varphi \le \theta_1$

  The segment cannot block up the roadway.

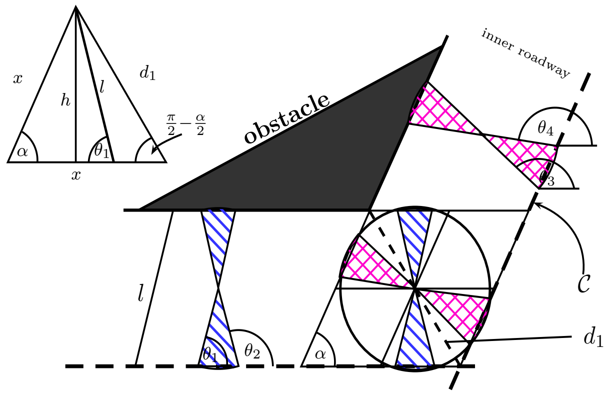

Figure 6: The inner roadway is blocked up, $h \le l \le x$ and $l \le d_1$

- $\theta_1 \le \varphi \le \theta_2$

  Figure 7 shows the admissible domain $\mathcal{C}_0^2(\varphi)$, consisting of 4 parts. The inner roadway can be blocked up by the segment, if its barycenter lies either in one of the parallelograms $\Gamma_0^2(\varphi)$ or in one of the triangles $\Delta_0^2(\varphi)$. The parallelogram has width $A-2x$ and height $l\sin\varphi - h$, the triangle has width $l\frac{\sin(\varphi-\alpha)}{\sin\alpha} - h\frac{\sin(\varphi-\alpha)}{\sin\alpha\,\sin\varphi}$ and height $l\sin\varphi - h$.

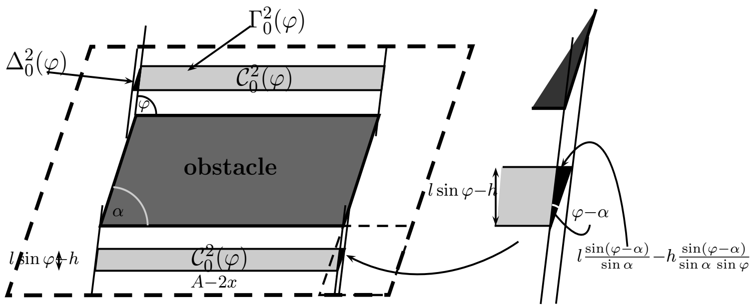

Figure 7: $\mathcal{C}_0^2(\varphi)$

We get

$$area(\mathcal{C}_0^2(\varphi)) = 2\,area(\Gamma_0^2(\varphi)) + 2\,area(\Delta_0^2(\varphi)) = 2\,Al\sin\varphi - 2\,Ah - 4\,lx\sin\varphi$$
$$+4\,hx + 2\,lx\sin(\alpha-\varphi) + hx\frac{\sin(\varphi-\alpha)}{\sin\varphi} + l^2\left((\cot\alpha)\sin^2\varphi - \sin\varphi\cos\varphi\right).$$

- $\theta_2 \le \varphi \le \theta_3 := \alpha + \arcsin\frac{h}{l}$

  The segment cannot block up the roadway.

- $\theta_3 \le \varphi \le \theta_4 := \pi + \alpha - \arcsin\frac{h}{l}$ We have

$$\begin{aligned} area(\mathcal{C}_0^4(\varphi)) &= 2\,(B-2\,x)\,l\sin(\varphi-\alpha) - 2\,h\,(B-2\,x) \\ &\quad + l^2\sin^2\varphi\cot\alpha - l^2\sin\varphi\cos\varphi - 2\,lx\sin\varphi + hx\frac{\sin\varphi}{\sin(\varphi-\alpha)}. \end{aligned}$$

Thus

$$\int_{\alpha+\arcsin\frac{h}{l}}^{\pi+\alpha-\arcsin\frac{h}{l}} area(\mathcal{C}_0^4(\varphi))d\varphi = 4(B-2\,x)\sqrt{l^2-h^2} + 4h(B-2\,x)\arcsin\tfrac{h}{l}$$
$$-2\,\pi h(B-2x) + l^2\left(\tfrac{1}{2}\pi - \arcsin\tfrac{h}{l}\right)\cot\alpha - 3\,x\sqrt{l^2-h^2}\cos\alpha + hx\left(\pi - 2\,\arcsin\tfrac{h}{l}\right)\cos\alpha.$$

- $\theta_4 \le \varphi \le \pi$

  The segment cannot block up the roadway.

In this way we have demonstrated the following:

Proposition 3 *If $h \le l \le x$ and $l \le d_1 = 2x\sin\frac{\alpha}{2}$, the probability p that a segment of constant length l, uniformly distributed in a bounded region of the plane,* ***blocks up the inner roadway****, is*

$$p_s \;=\; \frac{1}{\pi AB\sin\alpha}\Big(\left(\pi - 2\,\arcsin\tfrac{h}{l}\right)l^2\cot\alpha + 4\,(A+B-4\,x)\sqrt{l^2-h^2} - 2\,\pi h\,(A+B-4\,x)$$
$$+2\,\pi hx\cos\alpha + 4\,h\,(A+B-4\,x)\arcsin\tfrac{h}{l} - 4\,hx(\cos\alpha)\arcsin\tfrac{h}{l} - 6\,x\sqrt{l^2-h^2}\cos\alpha\Big).$$

Let be $\mathbf{h \le l \le x}$ and $\mathbf{l \ge d_1}$

Let be $l \ge d_1$. Figure 8 shows the different cases.

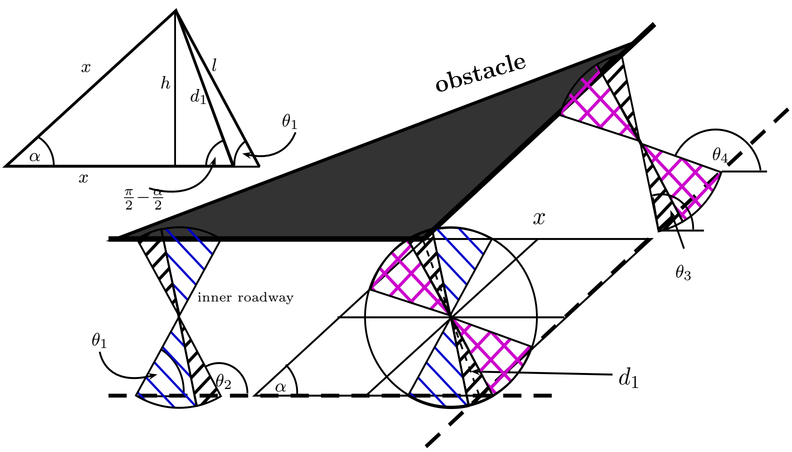

Figure 8: $h \le l \le x$ and $l \ge d_1$

Let be again $\theta_1 = \arcsin\frac{h}{l}$, $\theta_2 = \pi - \arcsin\frac{h}{l}$, $\theta_3 = \alpha + \arcsin\frac{h}{l}$, $\theta_4 = \pi + \alpha - \arcsin\frac{h}{l}$.

Lemma 2 *In this case $l \ge d_1$ we have a new order of inequalities*

$$0 \le \theta_1 \le \theta_3 \le \theta_2 \le \theta_4 \le \pi.$$

It is

$$\alpha + \theta_1 = \theta_3 \leq \theta_2 = \pi - \theta_1 \Longleftrightarrow \theta_1 \leq \frac{\pi}{2} - \frac{\alpha}{2}.$$

The most interesting case is

- $\theta_3 \leq \varphi \leq \theta_2 = \pi - \arcsin \frac{h}{l}$
  The admissible domain $\mathcal{C}_1^3(\varphi)$ is shown in figure 10 and figure 9 depending whether $\varphi \leq \frac{\pi+\alpha}{2}$ or $\varphi \geq \frac{\pi+\alpha}{2}$.

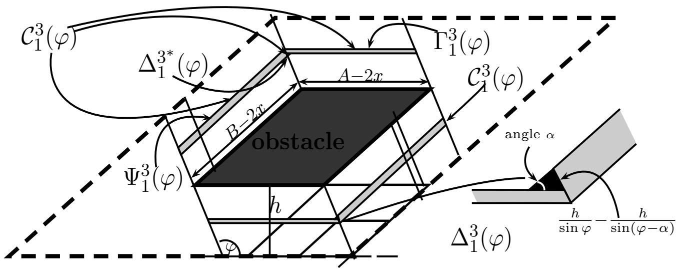

Figure 9: $\mathcal{C}_1^3(\varphi)$ for $\varphi \geq \frac{\pi+\alpha}{2}$ consists of 6 polygons

The admissible domain $\mathcal{C}_1^3(\varphi)$ consists of 6 polygons: there are two parallelograms $\Gamma_1^3(\varphi)$ with width $A-2x$ and height $l\sin\varphi - h$, two parallelograms $\Psi_1^3(\varphi)$ with width $B-2x$ and height $l\sin(\varphi-\alpha)-h$ and two triangles $\Delta_1^3(\varphi)$ and $\Delta_1^{3^*}(\varphi)$, having both the same area. For $\varphi \geq \frac{\pi+\alpha}{2}$ we have $\Delta_1^3(\varphi)$ with side $\frac{h}{\sin\varphi} - \frac{h}{\sin(\varphi-\alpha)}$ and angle α. We obtain

$$area(\Delta_1^3(\varphi)) = area(\Delta_1^{3^*}(\varphi)) = \frac{1}{2}\left(hx\cos\alpha - 2hx - h^2\cot\varphi + hx\frac{\sin\varphi}{\sin(\varphi-\alpha)}\right).$$

In case of $\varphi \leq \frac{\pi+\alpha}{2}$ the admissible domain $\mathcal{C}_1^3(\varphi)$ seems to be different; but it can be calculated by the same algebraic formula. Hence we get

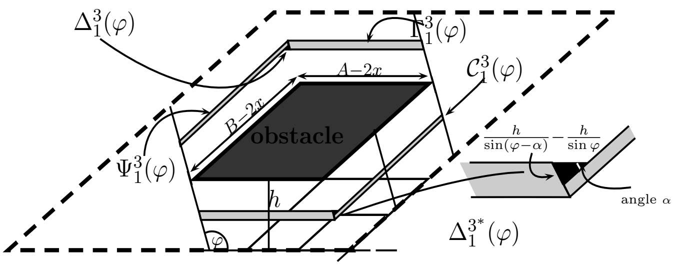

Figure 10: $\mathcal{C}_1^3(\varphi)$ for $\varphi \leq \frac{\pi+\alpha}{2}$ consists of 6 polygons

$$\begin{aligned} area(\mathcal{C}_1^3(\varphi)) &= 2\,area(\Gamma_1^3(\varphi)) + 2\,area(\Psi_1^3(\varphi)) + area(\Delta_1^3(\varphi)) + area(\Delta_1^{3^*}(\varphi)) \\ &= 2\,Al\sin\varphi - 2\,Ah + 6\,hx - 4\,lx\sin\varphi + hx\cos\alpha + \\ &\quad\; 2\,(B-2x)l\sin(\varphi-\alpha) - 2\,Bh - h^2\cot\varphi + hx\frac{\sin\varphi}{\sin(\varphi-\alpha)}. \end{aligned}$$

Integration leads to

$$\int_{\alpha+\arcsin\frac{h}{l}}^{\pi-\arcsin\frac{h}{l}} area(\mathcal{C}_1^3(\varphi))d\varphi = 4\,h(A+B-3x)\arcsin\tfrac{h}{l} + 2\,(\alpha-\pi)h(A+B-3x)$$
$$+2\,(\pi-\alpha)hx\cos\alpha - 4\,hx\,(\cos\alpha)\arcsin\tfrac{h}{l} + 2\,(A+B-4x)\sqrt{l^2-h^2}(1+\cos\alpha)$$
$$-2\,h(A+B-4x)\sin\alpha + 2\,h^2\left(\ln\!\left(\sqrt{l^2-h^2}\sin\alpha + h\cos\alpha\right) - \ln(h)\right).$$

In this way we have demonstrated the following:

Proposition 4 *Let be $h \leq l \leq x$ and $l \geq d_1$. The probability p that a segment of constant length l, uniformly distributed in a bounded region of the plane,* ***blocks up the inner roadway****, is*

$$p_s = \frac{1}{\pi AB\sin\alpha}\Bigg(\Big(\alpha\cot\alpha - 1\Big)l^2 - 2\,\pi h\Big(A+B-3\,x\Big) - 2\,h^2 + 4\,\Big(A+B-3\,x\Big)\sqrt{l^2-h^2}$$
$$+4\,h\Big(A+B-3\,x-x\cos\alpha\Big)\arcsin\tfrac{h}{l} - 4\,x\sqrt{l^2-h^2}\cos\alpha + 2\,\alpha hx + 2\,\pi hx\cos\alpha\Bigg).$$

In this paper we have seen some problems of Buffon type for the lattice of parallelograms with obstacles. In [3], these problems are also proofed for other cases, especially for larger segments. There are some other problems to solve regarding this lattice, e.g. the probability that the segment blocks up the outer roadway or even both roadways which is equivalent to the probability that a segment hits 2 obstacles simultaneously.

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PERDUTI E IN RITARDO, QUALE ORIENTAMENTO IN UN OTTICA DI FIDELIZZAZIONE ?

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1. Una premessa

Ormai la strada intrapresa dal Ministero dell'Università e della Ricerca di comune accordo con il Ministero dell'Economia e delle Finanze[1] è chiara, e seppure riferita ad una programmazione di breve-medio periodo ed ispirata ad un concetto di gradualità nel pieno rispetto dell'autonomia universitaria, non lascia spazio a dubbi e si afferma come scelta irreversibile. In sostanza gli Atenei, in un prossimo futuro, potranno esercitare la propria autonomia ma il loro operato sarà sottoposto a verifica. La scelta di dare il tempo agli Atenei di adeguarsi ed organizzarsi trova una motivazione più che valida nel fatto che, come sottolineato anche dal "Sole 24 ore" (23 luglio 2007), "se si applicassero tout court gli indicatori i conti di alcuni atenei subirebbero un colpo probabilmente mortale". Infatti la situazione attuale presenta marcate differenze tra i vari atenei (ad esempio: Università finanziate in eccesso, fino al 36%, e Università finanziate per difetto ,fino al 43,1%.). Proprio il nostro Ateneo è quello che subirebbe il taglio più duro: - 36.2% (% FFO assegnato sul totale = 2,63 ; % FFO teorico da modello 2006 = 1,68).

Questo in sintesi significa che, anche se sino ad oggi i Fondi Finanziamento Ordinario (FFO) sono stati ripartiti quasi esclusivamente sulla base delle quote storiche di spesa, gli atenei dovranno essere responsabili delle proprie decisioni autonome. Quindi fermo restando il rispetto del vincolo del 90% del FFO saranno previste incentivazioni per gli atenei sottofinanziati e per quelli che si dimostreranno virtuosi nella gestione delle politiche di bilancio, del personale, dell'offerta didattica e della produzione scientifica[2]. La valutazione sarà a carico dal Comitato Nazionale per la Valutazione del Sistema Universitario (CNVSU) e del Comitato di indirizzo per la valutazione della ricerca (CIVR) sino a quando l'Agenzia nazionale di valutazione del sistema universitario e della ricerca (ANVUR) non sarà pienamente operativa. Proprio l'ANVUR avrà il compito di segnalare la quota non consolidabile del FFO.

Per quello che riguarda la didattica, l'attuale modello del CNSVU, reso pubblico nel 2004 e modificato nel 2005, tiene conto del numero degli iscritti (full time e part time) agli anni successivi al primo, dei CFU (Crediti Formativi Universitari) guadagnati dagli iscritti in n+1 anni di corso (n = durata legale del corso) e del numero di laureati ponderati con un coefficiente che tiene conto del tempo impiegato per laurearsi. Nel medio periodo, però, entreranno in gioco altri indicatori legati ai riscontri occupazionali, alla provenienza geografica degli iscritti, alla provenienza universitaria degli iscritti alle lauree magistrali e alla durata del corso di studi specialistico (per approfondimenti si veda: " Patto per l'Università e la Ricerca ", Conferenza Stampa, Palazzo Chigi, 2 agosto 2007). Appare chiaro, quindi, che d'ora in poi tutti gli atenei, non solo, dovranno immatricolare un numero elevato di studenti ma dovranno far si che questi si iscrivano agli anni successivi, acquisendo il maggior numero di crediti, arrivando alla laurea entro il primo anno fuori corso. Inoltre meglio se gli studenti provengono da altre regioni e successivamente alla laurea trovano facilmente lavoro. E' evidente, dunque, che negli intenti del Ministero ci sia la volontà, non solo, di legare i finanziamenti alle "performance" degli Atenei, ma anche di rendere gli stessi Atenei responsabili del successo o dell'insuccesso delle scelte degli studenti che partecipano al processo formativo.

2. Il contesto generale

Il presente paragrafo è reperibile nella versione integrale del paper sul sito dell'Ufficio Statistico della Facoltà di Economia http://ww2.unime.it/fac_economia/docenti_fac/ricca/uffstat/pubblicazioni.htm

3. Il "drop out"

Negli ultimi anni, il fenomeno della dispersione e dell'abbandono è stato oggetto di particolari attenzioni. Esiste infatti, sull'argomento, una vasta letteratura[3] sia europea che americana che trova le sue radici nei modelli: "student integration" di Tinto[4] e "student attrition" di Bean[5]. Proprio grazie alle ricerche compiute nel passato, oggi si è arrivati ad una definizione univoca dal punto di vista formale. Distinguiamo, infatti, gli *abbandoni impliciti,* viene definita così la condizione degli studenti che

[1] "Patto per l'Università e la Ricerca", Conferenza Stampa, Palazzo Chigi, 2 agosto 2007

[2] Il decreto del MIUR n.362/2007 prevede che le risorse del Fondo per la Programmazione 2007-2009 siano ripartite in proporzione al peso calcolato con il modello CNSVU. "Già per il 2008 una quota del 5% del FFO va ripartita tra le Università non soggette a piani di risanamento sulla base della formula CNVSU

iscritti ad un anno accademico non rinnovano la loro iscrizione all'anno accademico successivo, e gli *abbandoni espliciti* ovverosia formalizzati, a questa categoria appartengono tutti coloro che hanno presentato istanza di rinuncia agli studi e gli studenti trasferiti ad altro Corso di Laurea, Facoltà o Ateneo[6]. Bisogna rilevare, invece, che per ciò che riguarda le motivazioni/cause che concorrono a determinare l'abbandono esiste una pluralità di aspetti riconducibili a tre classi categoriali[7] :

1. fattori di sistema
   - variabili connesse alle caratteristiche del percorso formativo individuale
2. fattori individuali
   - caratteristiche dello studente
   - livello di competenze dei soggetti in entrata
   - livello di confidenza/fiducia personale
   - capacità di gestione del tempo
   - scelte e decisioni di carriera
   - integrazione nel sistema universitario
3. fattori organizzativi
   - corso di studi
   - qualità dell'insegnamento
   - sistemi di sostegno consulenziale
   - modalità di reclutamento e selezione
   - cultura organizzativa

Inoltre si deve considerare che il fenomeno del drop out assume valenze e spiegazioni diverse a seconda dell'interpretazione che ne danno i diversi "stakeholder". Questo unito alla complessità e molteplicità degli aspetti chiamati in causa rende sfocato e quindi di difficile interpretazione il fenomeno stesso.

Si potrebbe concludere, quindi, che sarebbe più opportuno, ai fini di una pianificazione di interventi correttivi e/o di contenimento, considerare il drop out in funzione della tipologia di intervento. Ciò porta a distinguere due differenti punti di vista: quello tipico delle azioni di orientamento del "recupero" o del "salvataggio" e quello che fa riferimento alle strategie di fidelizzazione della "prevenzione" o del "contenimento". Quanto detto evidenzia che, probabilmente, il fenomeno del drop out, come quello del ritardo negli studi, non può essere spiegato solamente attraverso fattori individuali ma sarebbe necessario valutare e pesare, in funzione della strategia di intervento prescelta, l'importanza dell'influenza di altri fattori quali quello organizzativo e di sistema.

4. Il Fenomeno del "drop out" in Italia

Il presente paragrafo è reperibile nella versione integrale del paper sul sito dell'Ufficio Statistico della Facoltà di Economia http://ww2.unime.it/fac_economia/docenti_fac/ricca/uffstat/pubblicazioni.htm

5. L' indagine "MECAB"

A seguito dell'enorme importanza che ha assunto, come ampiamente dimostrato nelle righe precedenti, l'informazione sia a fini valutativi che conoscitivi orientati al problem solving e al decision making, la Facoltà di Economia ha istituito l'Ufficio Statistico di Facoltà[8] con l'intento non solo, di tenere sotto controllo il valore degli indicatori ufficiali, ma anche con l'obiettivo di approfondire alcuni temi ritenuti importanti nell'ottica del miglioramento continuo. L'Ufficio Statistico di Facoltà ha ereditato due indagini[9], "QUNI" e "STUDEME", svolte sugli universi degli studenti regolari e fuori corso, nell'ambito delle quali si metteva in luce la notevole incidenza del fenomeno degli abbandoni (impliciti/espliciti). Da questa evidenza e consci del fatto che il giudizio sui livelli di soddisfazione

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[3] Esistono sull'argomento numerosissime ricerche sviluppate in contesti diversi. Essendo impossibile citarle tutte si veda, ad esempio, tra gli altri: Trivellato (1978), Bernardi (1999), Denti e Schizzerotto (2002), Tanucci, Lo Presti, Spagnoli e Cerchi (2006), Cingano e Cipollone (2007)

[4] TINTO V.(1975), .Drop out from higher education: A theoretical synthesis of recent research. Review of Educational Research,

[5] BEAN J.P.(1980) .Drop out sand turn over: The synthesis and test of a causal model of student attrition. Research in Higher Education

BEAN J.P. , METZNER B.S.(1985), A conceptual model of non traditional undergraduate student attrition. Review of Educational Research

[6] In questo senso bisogna chiarire che per ciò che riguarda i trasferimenti in uscita, questi sono considerati differentemente a seconda del punto di vista. Nel caso di questo lavoro, il punto di vista è quello della Facoltà di Economia dell'Ateneo di Messina quindi verranno considerati abbandoni espliciti i trasferimenti ad altre Facoltà o Università.

[7] Per approfondimenti sui concetti si veda : Orientamento e carriera universitaria, (2006) a cura di Fasanella A. e Tanucci G., Franco Angeli (Cap. 2, Par. 5, Pag. 44)

[8] http://www.economia.unime.it/index.php?option=com_content&task=view&id=130&Itemid=122

[9] Riflessioni su alcuni aspetti quantitativi e qualitativi della Facoltà di Economia dell'Università degli Studi di Messina: la indagini STUDEME e QUNI, B. Ricca e M. Monastero, Annali della Facoltà di Economia dell'Università di Messina, 2003

espressi dagli studenti frequentanti è incompleto se non vengono considerati anche coloro che non hanno frequentato o che sono stati poco presenti e che successivamente hanno abbandonato, è nata l'idea di avviare un monitoraggio sulla popolazione degli immatricolati che non si sono iscritti al 2° anno dello stesso corso di Laurea. Le analisi svolte nell'ambito di questa indagine sperimentale, denominata "MECAB" tendono ad individuare metodologie semplici, ma ragionevolmente precise, per lo studio e il monitoraggio dei fenomeni dell'abbandono e dell'irregolarità degli studi nella Facoltà di Economia dell'Università degli Studi di Messina. In particolare ci si è prefissi, come già fatto da altri autori[10], di mostrare come si possono trasformare i dati amministrativi contenuti nell'archivio della segreteria degli studenti in informazioni idonee al governo e alla valutazione delle attività formative. In secondo luogo, si cercherà di porre in luce come, attraverso procedure di rilevazione abbastanza agevoli , si riescano a raccogliere dati affidabili circa le motivazioni sottostanti al comportamento di abbandono. La ragione per cui l'indagine ferma la propria attenzione sull'interruzione prematura degli studi universitari risiede nell'ovvia considerazione che, sotto il profilo sostanziale, essa rappresenta importante fenomeno di dispersione delle risorse collettive di capitale economico e di capitale umano, mentre, sotto il profilo tecnico, essa si configura come cruciale indicatore di efficacia dell'attività formativa degli atenei.

Nel conto delle motivazioni sottese alla progettazione dell'indagine va, poi, messo il convincimento che, a dispetto di opinioni assai diffuse, le università non possano essere considerate le uniche responsabili di tutti gli abbandoni e che, in non pochi casi, essi debbano, al contrario, essere fatti risalire a scelte individuali, o a vincoli strutturali, derivanti dai modi di funzionamento della società, sui quali poco le singole università possono incidere. Ma è proprio per cercare di capire in quale misura gli atenei siano causa degli abbandoni che, preventivamente, diventa necessario conoscere le dimensioni complessive del fenomeno e le motivazioni che vi soggiacciono. Pur trattandosi di un'iniziativa limitata alle non iscrizioni tra il primo e secondo anno della nostra Facoltà, dobbiamo sottolineare che le analisi che saranno svolte nel corso della ricerca e le procedure impiegate per effettuarle debbano essere suscettibili, dopo gli opportuni adattamenti del caso, di essere replicate anche da altre realtà della nostra Università.

Dopo una attenta analisi delle indagini svolte da altri Atenei si è deciso di effettuare una valutazione complessiva del fenomeno attraverso i dati delle segreterie[11] e successivamente una indagine campionaria in due step : il primo postale che prevedeva l'invio di una busta contenente una lettera di presentazione, il questionario ed una altra busta preaffrancata riportante l'indirizzo della Facoltà di Economia; il secondo telefonico per un eventuale recupero dei dati mancanti e per raggiungere un certo livello di affidabilità dei risultati.

5.1 Aspetti metodologici dell'indagine

Il progetto "MECAB" è stato avviato, per iniziativa dell'Ufficio Statistico delle Facoltà di Economia di comune accordo con la Presidenza della Facoltà, nell'anno accademico 2005-2006. In questo anno accademico, con la collaborazione delle le segreterie della Facoltà, è iniziata una prima ricognizione sugli abbandoni impliciti ed espliciti delle lauree di primo livello. Nell'anno accademico successivo si è passati, con l'autorizzazione dell'amministrazione dell'Ateneo, alla fase operativa. L'indagine è di tipo longitudinale e prevede una descrizione iniziale dell'impatto del fenomeno attraverso l'elaborazione dell'archivio fornito dal Centro di Calcolo di Ateneo. Successivamente si è fatto riferimento agli immatricolati della Facoltà di Economia e lo stesso archivio è stato impiegato, dopo un opportuna codifica dei dati[12], allo scopo di effettuare una analisi più approfondita mediante l'analisi di regressione logistica. Questo "è un metodo per la stima della funzione di regressione che meglio collega la probabilità del possesso di un attributo dicotomico con un insieme di variabili esplicative"[13]

In questo caso la variabile dipendente è rappresentata dal valore 1 se lo studente considerato si è riscritto all'anno successivo dello stesso corso di laurea e dal valore 0 se non ha rinnovato l'iscrizione. Questa metodologia di analisi permetterà di individuare le probabilità di rischio di tale scelta in funzione di una o più variabili esplicative (sesso, età, residenza, esami sostenuti). Per quello che invece riguarda l'indagine campionaria, gli elenchi dei non iscritti sono stati forniti, come già detto, dalle

[10] Tra gli altri si veda: A. Schizzerotto, 2002, Dinamiche e motivazioni dell'abbandono degli studi universitari: l'esperienza di Milano-Bicocca, pubblicato in "Valutazione dell'Università accreditamento del processo, misurazione del prodotto", Franco Angeli

[11] A questo proposito si desidera ringraziare il Centro di Calcolo di Ateneo e soprattutto le Segreterie della Facoltà di Economia per la disponibilità e la sensibilità dimostrata. In particolare per il completamento dell'indagine attraverso metodologia CATI si ringraziano i responsabili delle segreterie A. Perdichizzi e A. De Maria

[12] Ai fini dell'analisi le variabili sono state dicotomizzate (ad esempio variabile zona geografica di provenienza può essere dicotomizzata dividendo le unità, componenti il campione di riferimento, in appartenenti alla provincia di Messina e non appartenenti alla provincia di Messina

[13] L. Fabbris, "Statistica Multivariata", 1997, McGraw-Hill

segreterie. L'acquisizione delle informazioni è avvenuta per mezzo di un questionario[14] che è stato inviato per posta[15] a coloro che si erano immatricolati nell'Anno Accademico 2005-2006 e che in quello successivo non avevano rinnovato l'iscrizione allo stesso corso di laurea. In seguito il numero dei questionari "ritornati" è stato integrato mediante interviste telefoniche per raggiungere la significatività campionaria minima desiderata. In particolare il piano di campionamento è stato stabilito a posteriori, verificando, cioè, se il numero delle interviste effettuate potesse costituire la base per formare un campione[16] "quasi-casuale" , pur nella consapevolezza che "se le variabili da rilevare nell'indagine sono diverse, a rigore si dovrebbe definire un campione di numerosità appropriata per ciascuna variabile"[17]. Secondo la letteratura esistente[21], la dimensione ottimale di un campione casuale semplice senza reinserimento (*n'*) quando ci si riferisce a frequenze assolute o relative di certi attributi di una popolazione è determinabile mediante[18] :

$$n = \frac{S^2 k_{\alpha/2}^2}{e^2} \qquad \Longrightarrow \qquad n' = \frac{n}{1 + \frac{n}{N}}$$

dove, nel caso di frequenze, S^2 è uguale a 0,25 e rappresenta la varianza nella condizione più svantaggiosa, $k^2_{\alpha/2}$ viene dedotto dalle tavole della distribuzione normale standardizzata, e (errore tollerabile) corrisponde al livello di precisione desiderato ed N identifica la numerosità della popolazione. Nel nostro caso la popolazione di riferimento dei non iscritti al secondo anno e immatricolati l'anno precedente era composta da 369 unità, la risposta è stata del 25,2% che corrisponde a 93 questionari restituiti. Un campione casuale semplice con queste caratteristiche si configura con un livello di confidenza del 95% ed un errore tollerabile del 8,8%.

5.2 Il contesto: il Drop out nell'Ateneo Messinese

Il presente paragrafo è reperibile nella versione integrale del paper sul sito dell'Ufficio Statistico della Facoltà di Economia
http://ww2.unime.it/fac_economia/docenti_fac/ricca/uffstat/pubblicazioni.htm

5.3 La Facoltà di Economia : attrazione geografica, "fuoricorsismo", produttività, drop out

La Facoltà di Economia di Messina, come tutto il resto della realtà universitaria in Italia, sta cercando di adeguarsi ai veloci cambiamenti di questi ultimi anni. Infatti sono innumerevoli i mutamenti imposti dal nuovo ordinamento[19], dalla valutazione universitaria, dalla internazionalizzazione, dalle nuove esigenze di mercato, dalle innovazioni tecnologiche e dalla apertura di nuove sedi universitarie in zone limitrofe.

Ad oggi, infatti, sono stati risolti alcuni dei problemi che in passato sono stati oggetto di attenzione. Attualmente la Facoltà può contare su degli ampi e rinnovati locali, su una attrezzatura multimediale per la didattica di cui sono dotate tutte le aule, su due aule di informatica con attrezzature all'avanguardia, su un laboratorio linguistico multimediale, su una biblioteca con sala multimediale per la consultazione e su un settore stage e tirocini in piena espansione. Inoltre l'offerta didattica è certamente di ampio respiro comprendendo 7 lauree di primo livello e 6 di secondo livello. Senza contare i master e i dottorati organizzati dalla Facoltà o da strutture legate alla stessa Facoltà. Oggi dal punto di vista numerico, rispetto all'anno accademico 1998-99, la Facoltà di Economia ha subito (Tab. 11) una diminuzione del numero degli iscritti del 55.2%, dei fuori corso del 62.8% e delle immatricolazioni del 20.5%. Comunque, si deve tenere conto che certamente negli anni ha influito la concorrenza dei nuovi Atenei nati in zone limitrofe alla provincia messinese. Infatti nell'A.A. 1998-99 il 65.75% degli iscritti era residente fuori dalla provincia di Messina, contro il 45.41% nell'A.A. 2006-07. Se poi si guarda la composizione degli iscritti al nuovo ordinamento, in quest'ultimo anno accademico, si nota che i residenti fuori dalla provincia sono il 40.34% e tra gli immatricolati solo il 36.49% (nell'anno accademico 1998-99 gli immatricolati provenienti da una altra provincia erano il

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[14] Si ringraziano i colleghi A. Schizzerotto e F.Denti dell'Ateneo di Milano-Bicocca che, avendo svolto prima di noi una indagine sugli abbandoni del loro Ateneo, ci hanno fornito il loro questionario. Questo ci ha permesso di inserire, nel nostro questionario, alcune domande analoghe a quelle da loro proposte dandoci così la possibilità di confrontare due realtà universitarie così diverse.

[15] Si deve fare notare che ai fini della selezione campionaria sarebbe stato più corretto dal punto di vista procedurale intervistare gli ex-studenti per telefono, selezionandoli in maniera casuale dall'elenco fornito dalle segreterie. Si è scelto di procedere in due step (postale/telefonico) per non pesare ulteriormente sul personale delle segreterie già oberato di lavoro.

[16] "I campioni quasi-casuali sono dei campioni casuali formati 'a caso', ma senza adottare delle regole rigorose tali da assicurare ad ogni unità la stessa probabilità di entrare a far parte del campione", De Cristofaro, "Rilevazioni Campionarie. Breve Introduzione", Clueb, Bologna, 1979

[17] Fabbris L., "L'indagine campionaria. Metodi, disegni e tecniche di campionamento", NIS, Roma, 1993

[18] Sull'argomento si veda anche : Appendice D della Norma Italiana per i Sistemi di gestione della qualità UNI 11098

[19] La riforma è stata introdotta con la legge 197/97 e regolamentata dal DM 509/99

58%). Più grave la situazione dal punto di vista regionale, infatti il 63.6% degli iscritti alle nuove lauree sono siciliani, il 34.3% calabresi e solo il 2% provengono da altre regioni d'Italia. Quindi, ricordando che il Ministero ha inserito l'indicatore di attrazione geografica tra quelli che concorreranno a formare il giudizio di valutazione, sarà necessario, per non perdere ulteriore terreno, operare in questo senso, ad esempio delocalizzando alcuni corsi di laurea, aumentando il numero di borse di studio a favore dei non residenti" o utilizzando lo strumento dei prestiti d'onore per i più meritevoli.

Tabella 11

Iscritti Facolta di Economia di Messina[20]				
A.A.	**immatricolati**	**In corso**	**Fuori corso**	**Totale**
1998-99	990	2337	5488	8815
1999-00	905	1651	5537	8093
2000-01	829	1567	4776	7172
2001-02	993	1375	4049	6417
2002-03	884	1572	3401	5857
2003-04	876	1671	2778	5325
2004-05	872	1143	2477	4492
2005-06	787	1066	2314	4167
2006-07	781	1122	2041	3944

Un altro dei problemi che, da sempre, ha afflitto la Facoltà è stato il fenomeno del "fuoricorsismo" (Tab.12), nell'A.A. 1998-99 i fuori corso rappresentavano il 62.26% del totale degli iscritti, oggi il 51.75%. La riduzione rispetto all'A.A. 1998-99 del numero dei fuori corso è stata del 62.81% a fronte di una diminuzione del numero totale degli iscritti del 52.25%.

Tabella 12

Fuoricorso Facolta di Economia di Messina			
A.A.	**Variazione percentuale rispetto all'A.A. 1998-99**	**Variazione percentuale rispetto all'A.A. precedente**	**% Fuori corso Sul totale degli iscritti**
1998-99			62.26
1999-00	0.89	0.89	68.42
2000-01	- 12.97	- 13.74	66.59
2001-02	- 26.22	- 15.22	63.10
2002-03	- 38.03	- 16.00	58.07
2003-04	- 49.38	- 18.32	52.17
2004-05	- 54.87	- 10.84	55.14
2005-06	- 57.84	- 6.58	55.53
2006-07	- 62.81	- 11.80	51.75

Tabella 13

Fuoricorso Facolta di Economia di Messina			
A.A.	**Nuovo Ordinamento**	**Vecchio Ordinamento**	**Totali**
1998-99		5488	5488
1999-00		5537	5537
2000-01		4776	4776
2001-02	1	4048	4049
2002-03	6	3395	3401
2003-	102	2676	2778

Tabella 14

Fuoricorso Facolta di Economia di Messina Percentuali sul totale dei fuoricorso			
A.A.	**Nuovo Ordinamento**	**Vecchio Ordinamento**	**Totali**
1998-99	0.00	100.00	100.00
1999-00	0.00	100.00	100.00
2000-01	0.00	100.00	100.00
2001-02	0.02	99.98	100.00
2002-03	0.18	99.82	100.00

Tutti i dati riportati nella tabella 11 sono di fonte MIUR eccetto i valori relativi all'anno accademico 2006-07 che sono ottenuti da nostre elaborazioni dei dati forniti dal Centro di Calcolo di Ateneo

04			
2004-05	595	1882	2477
2005-06	881	1433	2314
2006-07	995	1046	2041

2003-04	3.67	96.33	100.00
2004-05	24.02	75.98	100.00
2005-06	38.07	61.93	100.00
2006-07	48.75	51.25	100.00

Inoltre, come si può facilmente vedere (Tab. 14) ad oggi la popolazione dei fuori corso (52.25% del totale iscritti) risulta composta per il 48.75% da iscritti alle lauree triennali o specialistiche e per un 51.25% di iscritti alle "vecchie" lauree quadriennali. In buona sostanza i "fuoricorso" continuano a pesare troppo, anche se considerando che nell'anno accademico 2006-07 gli iscritti totali alle lauree di primo e di secondo livello risultavano essere 2898, cioè il 72.5% di tutti gli iscritti, i fuori corso di "nuovo ordinamento" incidevano sul totale per solo il 34.33% che è pur sempre un terzo della popolazione totale (Tab.15).

Tabella 15

Fuoricorso **Facolta di Economia di Messina** Lauree Triennali e Specialistiche A.A. 2006-07			
A.A.	**Fuoricorso**	**Totale iscritti**	**% Fuoricorso sul Totale iscritti**
2004-05	595	2610	22.79
2005-06	881	2734	32.22
2006-07	995[21]	2898	34.33

In questo senso, relativamente alle lauree di primo livello dell'anno accademico 2006-07, vale la pena approfondire la conoscenza della popolazione dei fuori corso. Nelle successive tabelle (Tab. 16 – 17 – 18) si evidenzia che nel 50% dei casi i percorsi di studi a più alto tasso di fuoricorso, rispetto al totale degli iscritti allo stesso corso, sono il corso di laurea di Economia e Commercio (43.42%) e quello di Economia Bancaria (37.34%). Nel 72% dei casi lo studente fuori corso è andato oltre la durata legale di uno o due anni e ha un età compresa tra i 23 e i 26 anni.
Le alte percentuali di fuoricorso sembrano suggerire che vi sia una certa difficoltà nel mantenersi in regola con gli esami. Infatti, da questo punto di vista è interessante osservare che il 58.72% degli studenti fuori corso ha acquisito al massimo 120 cfu e che solo il 22.99% è quasi prossimo alla laurea. Dalla lettura dei dati contenuti nella Tabella 19 si potrebbe intuire che gli iscritti alla Facoltà di Economia incontrano oggettive difficoltà nell'affrontare gli studi, ciò potrebbe portare a due tipi di conseguenze: il ritardo alla laurea con un ovvio abbassamento del tasso di laureati in regola o l'abbandono. Come si rileva nelle tabelle successive, in cui si è ricostruita la carriera universitaria degli immatricolati nell'anno accademico 2003-04, il fenomeno della dispersione è piuttosto consistente soprattutto nel passaggio tra 1° e 2° anno di corso (41.3%), inoltre escludendo i Laureati in corso (4.1%) il 56.3% degli immatricolati nell'anno accademico 2003-04 non ha rinnovato l'iscrizione in uno degli anni successivi al primo e quindi solo il 39.6% continua gli studi almeno sino al 1° anno fuoricorso. In ogni caso è sicuramente interessante evidenziare (Tab. 22) che il 62.5%, degli immatricolati nel 2003-04 e iscritti al 1° anno fuoricorso nel 2006-07, risulta in forte ritardo con gli studi avendo conseguito al massimo 120 cfu.

Tabella 16

Fuoricorso **Facolta di Economia di Messina** Lauree Triennali A.A. 2006-07			
Corso di Laurea	**Valori assoluti F.C.**	**Totale iscritti**	**Valori %**
Economia e commercio	221	509	43.42
Economia bancaria	267	715	37.34
Economia del turismo e dell'ambiente	200	661	30.26
Economia e amm.ne delle imprese	106	338	31.36
Economia aziendale	171	479	35.70
Totale complessivo	*965*	*2702*	*35.71*

Tabella 17

Fuoricorso **Facolta di Economia di Messina** Lauree Triennali A.A. 2006-07		
Anno F.C.	**Valori assoluti**	**Valori %**
1	393	40.73
2	311	32.23
3	255	26.42
4	6	0.62
Totale complessivo	*965*	*100.00*

Tabella 19

Tabella 18

[21] I 995 studenti fuoricorso risultano così suddivisi : 965 iscritti alle lauree di primo livello e 30 iscritti alle lauree specialistiche

Frequenze relative percentuali degli studenti Fuoricorso in base ai cfu acquisiti Facoltà di Economia di Messina Lauree Triennali A.S. 2006					
cfu acquisiti	**<=60**	**<=120**	**<=140**	**<=180**	**Tot.**
Economia e commercio	18.64	38.64	19.55	23.18	100.00
Economia bancaria	21.43	45.11	14.29	19.17	100.00
Economia del turismo e dell'ambiente	19.80	35.64	17.82	26.73	100.00
Economia e amm.ne delle imprese	19.42	33.01	15.53	32.04	100.00
Economia aziendale	20.48	35.54	25.30	18.67	100.00
Totale	***20.06***	***38.66***	***18.29***	***22.99***	***100.00***

Fuoricorso **Facoltà di Economia di Messina** Lauree Triennali A.A. 2006-07		
Classi di età	**Valori assoluti**	**Valori %**
<=22	53	5.49
23-24	484	50.16
25-26	260	26.94
27-28	84	8.70
29-30	36	3.73
>30	48	4.97
Totale complessivo	*965*	*100.00*

Tabella 20

Prosecuzioni e dispersioni della leva di immatricolati 2003-04 in base al Corso di Laurea prescelto al momento dell'immatricolazione							
		EC	**EBFA**	**ETA**	**EAI**	**EAZ**	**Facoltà**
A.A. 2003-04	**Immatricolati 2004**	**180**	**200**	**228**	**130**	**139**	**877**
A.A. 2004-05	**NON ISCRITTI AL 2° ANNO**	**76**	**68**	**92**	**67**	**59**	**362**
	di cui trasferiti altra Facoltà Stesso Ateneo	*1*		*1*		*2*	*4*
	di cui Abbandoni espliciti	*21*	*16*	*22*	*15*	*17*	*91*
	di cui Abbandoni impliciti	*54*	*52*	*69*	*52*	*40*	*267*
	ISCRITTI AL 2° ANNO	104	132	136	63	80	515
A.A. 2005-06	**NON 'ISCRITTI AL 3° ANNO**	**21**	**18**	**21**	**9**	**9**	**78**
	di cui Abbandoni espliciti	*6*	*7*	*9*	*2*	*4*	*28*
	di cui Abbandoni impliciti	*15*	*11*	*12*	*7*	*5*	*50*
	ISCRITTI AL 3° ANNO	83	114	115	54	71	437
A.A. 2006-07	***Laureati in corso***	***2***	***9***	***8***	***7***	***10***	***36***
	NON ISCRITTI AL 1° ANNO FC	**13**	**13**	**19**	**5**	**4**	**54**
	di cui Abbandoni espliciti	*4*	*4*	*2*	*1*	*2*	*13*
	di cui Abbandoni impliciti	*9*	*9*	*17*	*4*	*2*	*41*
	ISCRITTI AL 1° ANNO FC	**68**	**92**	**88**	**42**	**57**	**347**

Tabella 21

Prosecuzioni e dispersioni della leva di immatricolati 2003-04 in base al Corso di Laurea prescelto al momento dell'immatricolazione *(valori %)*							
		EC	**EBFA**	**ETA**	**EAI**	**EAZ**	**Facoltà**
A.A. 2003-04	**Immatricolati 2004**	**100**	**100**	**100**	**100**	**100**	**100**
A.A. 2004-05	**NON ISCRITTI AL 2° ANNO**	**42.2**	**34.0**	**40.4**	**51.5**	**42.4**	**41.3**
	di cui trasferiti	*1.3*	*0.0*	*1.1*	*0.0*	*3.4*	*1.1*
	di cui Abbandoni espliciti	*27.6*	*23.5*	*23.9*	*22.4*	*28.8*	*25.1*
	di cui Abbandoni impliciti	*71.1*	*76.5*	*75.0*	*77.6*	*67.8*	*73.8*
	ISCRITTI AL 2° ANNO	57.8	66.0	59.6	48.5	57.6	58.7
A.A. 2005-06	**NON 'ISCRITTI AL 3° ANNO**	**11.7**	**9.0**	**9.2**	**6.9**	**6.5**	**8.9**
	di cui Abbandoni espliciti	*28.6*	*38.9*	*42.9*	*22.2*	*44.4*	*35.9*
	di cui Abbandoni impliciti	*71.4*	*61.1*	*57.1*	*77.8*	*55.6*	*64.1*
	ISCRITTI AL 3° ANNO	46.1	57.0	50.4	41.5	51.1	49.8
A.A. 2006-07	***Laureati in corso***	***1.1***	***4.5***	***3.5***	***5.4***	***7.2***	***4.1***

NON ISCRITTI AL 1° ANNO FC	**7.2**	**6.5**	**8.3**	**3.8**	**2.9**	**6.2**
di cui Abbandoni espliciti	*30.8*	*30.8*	*10.5*	*20.0*	*50.0*	*24.1*
di cui Abbandoni impliciti	*69.2*	*69.2*	*89.5*	*80.0*	*50.0*	*75.9*
ISCRITTI AL 1° ANNO FC	**37.8**	**46.0**	**38.6**	**32.3**	**41.0**	**39.6**

Tabella 22

Crediti acquisiti dalla leva di immatricolati 2003-04 e iscritti al 1° anno fuoricorso
in base al Corso di Laurea prescelto al momento dell'immatricolazione
(valori %)

	EC	EBFA	ETA	EAI	EAZ	TOTALE
<= 60 cfu	17.6	22.8	21.6	26.2	24.6	22.2
<=120 cfu	35.3	52.2	40.9	38.1	28.1	40.3
<=150 cfu	32.4	14.1	22.7	23.8	36.8	24.8
<=180 cfu	13.2	9.8	14.8	11.9	8.8	11.8
Studenti a 0 cfu (posizione da chiarire)	*1.5*	*1.1*	*0.0*	*0.0*	*1.8*	*0.9*
	100	100	100	100	100	100

Come indicato nella Tabella 21 la maggior parte dei "drop out" si verifica tra il 1° e 2° anno di corso, questo induce ad una riflessione per meglio comprendere il fenomeno della dispersione subito dopo l'immatricolazione.
Osservando gli ultimi dati disponibili (Tab. 24) relativi agli immatricolati nell'anno accademico 2005-06 si nota che la percentuale di coloro che hanno abbandonato gli studi nel 2006-07 è leggermente aumentata, in quasi tutti i corsi di laurea, rispetto all'anno accademico 2004-05 e la propensione all'abbandono è più maschile (50.7%) che femminile (37.5%).

Tabella 23

Immatricolati A.A. 2005-06
che hanno o non hanno rinnovato l'iscrizione al 2° anno dello stesso C.L.
Valori assoluti

ISCRITTO AL 2° ANNO	EC	EBFA	ETA	EAI	EAZ	Totale
NON ISCRITTO	55	93	109	56	50	363
ISCRITTO	69	112	112	65	93	451
iscritti 1° anno altro C.L. stessa Facoltà	1			1	4	6
Totale immatricolati A.A. 2005-06	*125*	*205*	*221*	*122*	*147*	*820*

Tabella 24

Immatricolati A.A. 2005-06
che hanno o non hanno rinnovato l'iscrizione al 2° anno dello stesso C.L.
Valori assoluti

ISCRITTO AL 2° ANNO	EC	EBFA	ETA	EAI	EAZ	Totale
NON ISCRITTO	44.00	45.37	49.32	45.90	34.01	44.27
ISCRITTO	55.20	54.63	50.68	53.28	63.27	55.00
iscritti 1° anno altro C.L stessa facoltà	0.80	0.00	0.00	0.82	2.72	0.73
Totale immatricolati A.A. 2005-06	*100.00*	*100.00*	*100.00*	*100.00*	*100.00*	*100.00*

Tabella 25

Immatricolati A.A. 2005-06
che hanno o non hanno rinnovato l'iscrizione al 2° anno dello stesso C.L.
per sesso
Valori %

ISCRITTO AL 2° ANNO	F	M	Totale
NON ISCRITTO	37.5	50.7	45.0
ISCRITTO	62.3	48.1	55.0
iscritti 1° anno altro C.L. stessa Facoltà	0.3	1.2	0.7
Totale complessivo	*100.0*	*100.0*	*100.0*

Per quello che riguarda l'attrazione geografica la Facoltà esprime il 36.5% di immatricolati provenienti da "fuori regione" di cui, però, la quasi totalità dalla vicina Calabria (88.3% del totale dei residenti non siciliani). Inoltre, come si può verificare dalla tabella 26, i corsi di laurea che risultano più attraenti per i non residenti sono: Economia Aziendale (51%) ed Economia Bancaria (40.5). Si deve però notare che il corso di Economia del Turismo pur avendo la più bassa percentuale totale (28.1%) esprime, grazie alla sua delocalizzazione didattica, una forte attrazione per i residenti in altre regioni italiane (45.2% del totale dei residenti non siciliani).

Tabella 26

Immatricolati A.A. 2005-06
Per regione di residenza e per Corso di laurea
Valori %

Regione di residenza	**EC**	**EBFA**	**ETA**	**EAI**	**EAZ**	**Facolta**
SICILIA	64.8	59.5	71.9	71.3	49.0	63.5
ALTRE REGIONI D'ITALIA	35.2	40.5	28.1	28.7	51.0	36.5
di cui Calabria	*97.7*	*96.4*	*54.8*	*97.1*	*97.3*	*88.3*
di cui altre regioni	*2.3*	*3.6*	*45.2*	*2.9*	*2.7*	*11.7*

In ogni caso osservando i dati sugli abbandoni incrociati con la provincia di residenza (Tab. 27), ci si accorge che quest'ultima non sembra essere una causa predominante.

Tabella 27

Immatricolati A.A. 2005-06
che hanno o non hanno rinnovato l'iscrizione al 2° anno dello stesso C.L.
per provincia di residenza (*Valori %*)

Provincia/Regione di residenza	**NON ISCRITTO**	**ISCRITTO**	***iscritti 1° anno altro C.L. stessa Facolta***	**Totale**
REGIONE SICILIA	43.0	56.0	*1.0*	100.0
di cui Messina città	*43.3*	*38.7*	*80.0*	
di cui Messina provincia	*48.2*	*55.5*	*20.0*	
di cui altra provincia siciliana	*8.5*	*5.8*	*0.0*	
REGIONE CALABRIA	44.3	55.3	*0.4*	100.0
di cui Reggio Calabria città	*34.2*	*34.2*	*100.0*	
di cui Reggio Calabria provincia	*54.7*	*58.2*	*0.0*	
di cui altra provincia calabrese	*11.1*	*7.5*	*0.0*	
ALTRE REGIONI D'ITALIA	62.9	37.1	*0.0*	100.0

Di rilievo sembra essere, invece, la distinzione per età. Infatti, anche se gli immatricolati che hanno abbandonato gli studi nel 2006-07 hanno nel 62.8% dei casi, com'è ovvio aspettarsi, un età all'immatricolazione compresa tra 18 e 21 anni, è interessante notare che se si calcola la percentuale di abbandono relativamente alle classi di età (Tab. 27-28) si scopre che coloro che si immatricolano in età avanzata sono più portati a non rinnovare l'iscrizione al secondo anno di corso.

Tabella 28

Immatricolati A.A. 2005-06
che hanno o non hanno rinnovato l'iscrizione al 2° anno dello stesso C.L.
per classi di eta
Valori %

ISCRITTO AL 2° ANNO	**18-19**	**20-21**	**22-23**	**24-25**	**26-30**	**>30**	**Totale**
NON ISCRITTO	32.4	53.6	51.7	75.0	83.3	75.0	45.0
iscritto	*67.4*	*42.8*	*48.3*	*25.0*	*16.7*	*25.0*	*55.0*
iscritti 1° anno altro C.L. stessa Facoltà	*0.2*	*3.6*	*0.0*	*0.0*	*0.0*	*0.0*	*0.0*
Totale immatricolati A.A. 2005-06	*100.0*	*100.0*	*100.0*	*100.0*	*100.0*	*100.0*	*100.0*

Tabella 29

Percentuale di "abbandoni" tra il 1° e 2° anno
degli immatricolati nell'A.A. 2005-06
per classi di eta e corso di laurea
Valori %

c.l.	**18-19**	**20-21**	**22-23**	**24-25**	**26-30**	**>30**

EC	31.4	40.0	77.8	100.0	100.0	100.0
EBFA	29.1	76.7	22.2	71.4	92.3	100.0
ETA	38.6	46.7	50.0	80.0	77.8	57.5
EAI	33.8	50.0	54.5	66.7	71.4	100.0
EAZ	26.7	55.6	60.0	50.0	66.7	80.0

Un dato che certamente risulta indicativo tra le cause di "abbandono" riguarda la produttività. Il Ministero calcola la produttività mediante l'indicatore studente equivalente (rapporto tra il numero di crediti effettivamente acquisiti e il numero massimo di crediti conseguibili in un anno). Nel caso degli immatricolati della leva 2005-06 i crediti conseguiti (Tab. 30) sono stati solo il 28.46% del totale.

Tabella 30

Studente Equivalente
(N° crediti conseguiiti / N° crediti conseguibili)
Relativo agli immatricolati 2005-06

C.L.	**Immatricolati 2005-06**	**N° crediti conseguibili nell'anno 2006**	**N° crediti conseguiiti nell'anno 2006**	**Studente equivalente *Valore %***
EC	125	7500	2200	29.33
EBFA	205	12300	3836	31.19
ETA	221	13260	2860	21.57
EAI	122	7320	1734	23.69
EAZ	147	8820	3374	38.25
Facoltà Economia	***820***	***49200***	***14004***	***28.46***

In ogni caso vale la pena evidenziare che il 41.59% degli immatricolati non ha sostenuto nemmeno un esame nell'anno solare 2006 e di questi l'87.10% non si è iscritto al secondo anno. Inoltre sarebbe utile approfondire le motivazioni legate all'abbandono relativamente ai corsi di laurea di Economia e Commercio ed Economia Aziendale , infatti disaggregando i "non iscritti" per corso di laurea ci si accorge che il 25 % degli immatricolati non iscritti al primo dei due corsi di laurea ed il 27.8% del secondo aveva dato almeno un esame.

Tabella 31

Immatricolati A.A. 2005-06
che hanno o non hanno rinnovato
l'iscrizione al 2° anno dello stesso C.L.
che hanno o non hanno dato esami
nell'anno solare 2006 (*Valori %)*

iscritto 2° anno	**non hanno dato esami**	**hanno dato esami**	***Totale***
non iscritto	80.49	19.51	*100.00*
iscritto	9.76	90.24	*100.00*
Totale complessivo	*41.59*	*58.41*	*100.00*

Tabella 32

Immatricolati A.A. 2005-06
che hanno o non hanno rinnovato
l'iscrizione al 2° anno dello stesso C.L.
che hanno o non hanno dato esami
nell'anno solare 2006 (*Valori %)*

iscritto 2° anno	**non hanno dato esami**	**hanno dato esami**
non iscritto	87.10	15.03
iscritto	12.90	84.97
Totale complessivo	*100.00*	*100.00*

Se poi si guarda la distribuzione dei crediti acquisiti da coloro che hanno rinnovato l'iscrizione (Tab.34) si nota che oltre la metà degli iscritti al secondo anno ha conseguito al massimo la metà dei crediti conseguibili. Questo ritardo, oltre a poter diventare una delle probabili cause di abbandono negli anni successivi al primo, avrà inevitabili conseguenze sulla prosecuzione degli studi e quindi inciderà sui parametri relativi ai fuoricorso e laureati in regola, che come detto nella parte iniziale di questo lavoro sono importanti ai fini dell'assegnazione di una quota del F.F.O.

Tabella 33

Non iscritti al secondo anno dell'A.A. 2006-07
che hanno o non hanno dato esami
per Corso di Laurea
Valori %

C.L.	**non hanno**	**hanno**

Tabella 34

Immatricolati che si sono iscritti al 2° anno
suddivisi per numero di crediti acquisiti nel 2006
Valori %

C.L.	**=0**	**<=10**	**<= 20**	**<=30**	**<=40**	**<=50**	**<=59**	**=60**	**Totale**

	dato esami	dato esami
EC	75.0	25.0
EBFA	82.8	17.2
ETA	83.5	16.5
EAI	84.2	15.8
EAZ	72.2	27.8

EC	14.1	8.5	11.3	18.3	12.7	23.9	8.5	2.8	100.0
EBFA	6.2	9.7	6.2	19.5	23.9	23.9	9.7	0.9	100.0
ETA	14.7	14.7	16.5	22.0	16.5	4.6	0.9	10.1	100.0
EAI	9.0	7.5	22.4	23.9	19.4	17.9	0.0	0.0	100.0
EAZ	8.4	3.2	9.5	13.7	28.4	30.5	2.1	4.2	100.0

Le probabilità di drop out sono certamente legate alla produttività più che ad ogni altra caratteristica fin qui esaminata. Infatti utilizzando l'analisi di regressione logistica[22] per stimare la probabilità di drop out (variabile dipendente) in funzione delle variabili esplicative: sesso, età, residenza, esami[23]; si desume che gli studenti che non hanno conseguito crediti hanno una propensione ad abbandonare gli studi, durante il primo anno, 37 volte maggiore di coloro che hanno dato almeno una materia.

Tabella 35

	Estimate	Standard Error	p-level	Odds ratio
SESSO (rif. Maschio)	-	-	-	-
Femmina	0.43	0.21	0.04	1.53
ETA' (rif. 18-19)	-	-	-	-
20-21	- 0.23	0.29	0.23	0.80
22-23	- 0.03	0.40	0.09	0.97
24-25	- 0.88	0.63	0.01	0.41
26-30	- 1.59	0.60	0.01	0.20
>30	- 1.99	0.47	0.00	0.14
RESIDENZA (rif. Messina città)	-	-	-	-
Messina provincia	0.59	0.28	0.03	1.80
Sicilia altra provincia	- 0.06	0.55	0.09	0.94
Reggio Calabria città	- 0.33	0.37	0.37	0.72
Reggio Calabria provincia	0.11	0.32	0.07	1.11
Calabria altra provincia	0.03	0.60	0.09	1.03
Altra regione italiana	1.02	0.57	0.08	2.77
ESAMI (rif. Ha sostenuto esami)	-	-	-	-
Non ha sostenuto esami	3.61	0.22	0.00	37.06
Const.B0	*- 2.03*	*0.30*	*0.00*	*0.13*

Il modello ci permette, inoltre, di verificare che sono più a rischio di abbandono le donne rispetto agli uomini, coloro che hanno un età all'immatricolazione tra i 22 e i 23 anni rispetto agli appartenenti alla classe 18-19 e i residenti in una regione diversa da Sicilia e Calabria rispetto ai "messinesi".

5.4 Le Motivazioni dell'abbandono : un indagine campionaria

Partendo dal presupposto che le motivazioni sottese all'abbandono degli studi non sono responsabilità esclusiva degli Atenei o delle singole Facoltà e, come già detto, gli archivi delle segreterie non contengono dati di tipo sociale si è voluto approfondire, attraverso un indagine campionaria, la conoscenza della popolazione dei non iscritti al secondo anno (A.A. 2006-07) e immatricolati l'anno precedente (A.A. 2005-06).

Come riportato nel paragrafo 5.1, di questo lavoro, la popolazione di riferimento era composta da 369 unità e la risposta è stata del 25,2% che corrisponde a 93 questionari restituiti. Un campione casuale semplice con queste caratteristiche si configura con un livello di confidenza del 95% ed un errore tollerabile del 8,8%.

[22] "L'analisi di regressione logistica è un caso speciale dell'analisi di regressione che trova applicazione quando la variabile dipendente è dicotomica o dicotomizzata", per approfondimenti si veda : L. Fabbris, Statistica multivariata … Fonte già citata

[23] Gli archivi delle segreterie non contengono dati di tipo sociale ma sarebbe interessante approfondire questa tipologia di analisi introducendo altre variabili quali : titolo di studio dei genitori, situazione lavorativa oltre a titolo di studio e voto alla maturità.

Dalle tabelle 36 e 37 si evince che il campione degli intervistati risulta proporzionalmente ben distribuito sia per sesso che per residenza. Come era ovvio aspettarsi, visti i dati dell'archivio segreterie, la maggior parte (67,3%) non aveva sostenuto nemmeno un esame (Tab. 38) e più o meno la metà di coloro che invece avevano affrontato qualche prova di verifica non era stato mai bocciato.

Tabella 36

Rispondenti per sesso e classi di età
Valori %

Classe di età / Sesso	Maschio	Femmina	Totale
18-19	40.74	59.26	100.00
20-21	41.46	58.54	100.00
>21	58.33	41.67	100.00
Totale complessivo	***45.65***	***54.35***	***100.00***

Tabella 37

Rispondenti per luogo di residenza
Valori %

Residenza	%
Messina città	21.74
Provincia di Messina	31.52
Altra provincia siciliana	8.70
Reggio Calabria città	8.70
Provincia di Reggio Calabria	19.57
Altra provincia calabrese	6.52
Altra regione italiana	3.26
Totale	***100.00***

Certamente di maggiore interesse risulta la posizione occupazionale degli immatricolati, infatti il 53.2% degli intervistati ha dichiarato di lavorare al momento dell'immatricolazione (Tab. 39). Ovviamente la posizione lavorativa costituisce un impedimento a dedicarsi agli studi a tempo pieno, anche se il 32.6% del totale occupati aveva un lavoro saltuario. Nel questionario, però, si chiedeva agli intervistati di dichiarare, anche, la propria situazione lavorativa al momento dell'intervista. Questo ha permesso, incrociando i dati relativi ai due differenti istanti temporali (Tab. 40), di scoprire che coloro che hanno dichiarato di avere una attività lavorativa al momento dell'intervista era nel 41.8% dei casi non occupato e che solo il 30.6% ha "perso" il lavoro (il 47% di questi aveva precedentemente un lavoro saltuario). Questo porta a ritenere, anche osservando i successivi risultati dell'intervista, che una delle principali cause di abbandono è sicuramente la posizione occupazionale durante gli studi.

Tabella 38

Rispondenti che hanno o non hanno dato esami
Valori %

Hanno dato esami ?	%
NO non hanno dato esami	67.39
SI hanno dato almeno un esame	32.61
di cui mai bocciato	*46.7*
di cui bocciato più volte	*10.0*
di cui bocciato una volta	*43.3*
Totale	***100.00***

Tabella 39

Rispondenti che nel periodo in cui sono stati iscritti avevano o non avevano un'attività lavorativa
Valori %

Avevano un attività lavorativa ?	%
NO	46.74
SI	53.26
di cui continuativa a part-time	*18.3*
di cui continuativa a tempo pieno	*48.9*
di cui saltuaria	*32.6*
Totale	***100.00***

Tabella 40

Rispondenti che nel periodo in cui sono stati iscritti e nel periodo dell'intervista hanno dichiarato di avere o non avere un'attività lavorativa
Valori %

		Durante gli studi		
		Non Lavorava	**Lavorava**	**Totale**
Attualmente	**Non Occupato**	58.14	30.61	43.48
	Occupato	41.86	69.39	56.52
	Totale	***100.00***	***100.00***	***100.00***

Infatti osservando la tabella 41 nella quale sono riepilogate percentualmente le cause dell'abbandono identificate dagli intervistati come motivazioni principali o parziali si evince che all'origine del drop out sembrano essere: la mancanza di tempo da dedicare allo studio, i motivi legati alla distanza da

casa o alla difficoltà negli spostamenti, l'organizzazione dei corsi che non teneva conto delle esigenze dei non frequentanti, l'aver trovato un lavoro e gli studi rivelatisi incompatibili con il lavoro svolto.
Era stato chiesto, inoltre, agli intervistati di dichiarare quale fosse tra tutte le motivazioni la più importante(Tab. 42). Anche in questo caso la cause principali sono: la mancanza di tempo da dedicare allo studio e l'aver trovato un lavoro a cui si aggiungono i motivi familiari e la distanza da casa.
Le risposte sembrano confermare, in prima analisi, che la motivazione principale di abbandono sia la posizione lavorativa associata ad una difficoltà a dedicare tempo agli studi.

Tabella 41

Motivazione sottesa alla scelta di abbandonare gli studi
Valori %

	Si	In parte	No
per mancanza di tempo da dedicare allo studio	9.0	7.9	2.5
per motivi legati alla distanza da casa o alla difficoltà negli spostamenti	7.0	4.9	3.2
perché l'organizzazione dei corsi non teneva conto delle esigenze dei non frequentanti	6.5	6.7	3.1
perché ha trovato un lavoro	6.5	6.7	3.1
perché gli studi si sono rivelati incompatibili con il lavoro che svolgeva	5.6	4.3	3.5
perché è diminuito il suo interesse per gli studi intrapresi	5.6	5.5	3.4
perché aveva il desiderio di rendersi indipendente dalla sua famiglia	4.8	4.3	3.6
perché si è accorto/a di avere sbagliato la scelta del corso di studio	4.8	4.3	3.6
per la scarsa disponibilità dei docenti	4.5	3.7	3.7
per la disorganizzazione dei corsi e delle lezioni	4.2	3.7	3.8
per la disorganizzazione degli uffici (segreterie, amministrazione)	4.2	3.7	3.8
perché ha scoperto di essere interessato ad altri tipi di studio	4.2	3.7	3.8
per la carenza di spazi per lo studio	3.9	4.3	3.8
per l'impossibilità di sostenere i costi degli studi (compresi trasporti e alloggio)	3.7	2.4	4.0
perché è rimasto/a deluso/a dall'esperienza universitaria	3.7	4.3	3.8
perché si reso/a conto di non avere più voglia di studiare	3.7	5.5	3.7
perché sceglierà o ha già scelto un ateneo più prestigioso	3.1	1.2	4.2
per motivi familiari o di salute (sua o di suoi familiari)	3.1	2.4	4.1
per la carenza di laboratori per le esercitazioni	2.5	1.8	4.3
perché il corso di studi si è rivelato troppo difficoltoso	2.5	4.9	4.0
per la bassa qualità dei servizi di accoglienza offerti (mense, attività ricreative)	2.0	4.9	4.1
perché si è accorto/a che il corso di studio era poco professionalizzante	1.1	2.4	4.5
perché la preparazione ricevuta nella scuola superiore si è rivelata insufficiente	1.1	3.7	4.4
perché gli studi si sono dimostrati poco rigorosi	0.8	1.8	4.6
Totale	***100***	***100***	***100***

Tabella 42

La Motivazione più importante che ha determinato la scelta di abbandonare gli studi
Valori %

per mancanza di tempo da dedicare allo studio	17.4
per motivi legati alla distanza da casa o alla difficoltà negli spostamenti	10.9
per motivi familiari o di salute (sua o di suoi familiari)	8.7
perché ha trovato un lavoro	8.7
perché ha scoperto di essere interessato ad altri tipi di studio	7.6
per la disorganizzazione dei corsi e delle lezioni	6.5

perché l'organizzazione dei corsi non teneva conto delle esigenze dei non frequentanti	6.5
per l'impossibilità di sostenere i costi degli studi (compresi trasporti e alloggio)	5.4
perché si reso/a conto di non avere più voglia di studiare	5.4
perché sceglierà o ha già scelto un ateneo più prestigioso	4.3
perché è diminuito il suo interesse per gli studi intrapresi	3.3
perché gli studi si sono rivelati incompatibili con il lavoro che svolgeva	3.3
perché si è accorto/a di avere sbagliato la scelta del corso di studio	3.3
per la scarsa disponibilità dei docenti	2.2
perché è rimasto/a deluso/a dall'esperienza universitaria	1.1
perché il corso di studi si è rivelato troppo difficoltoso	1.1
perché la preparazione ricevuta nella scuola superiore si è rivelata insufficiente	1.1

Ritenendo particolarmente importante stabilire quale sia, secondo il giudizio degli intervistati, il peso che ha avuto, sulla scelta di abbandonare gli studi, l'organizzazione sia didattica che di struttura della Facoltà, si è deciso di sintetizzare la matrice delle motivazioni date attraverso una tecnica di scaling multidimensionale a cui è stata associata successivamente a scopo solo confermativo una cluster analysis. Tale metodologia consente di individuare le dimensioni significative sottostanti ai dati che permettono di spiegare similarità o dissimilarità osservate tra gli stessi dati oggetto di studio[24]. In questo caso la tecnica è stata applicata alla matrice di distanza media assoluta, particolarmente adatta per variabili su scala ordinale. Come evidenziato dall'output grafico (Graf. 1) è stato possibile individuare 5 macro-indicatori che sintetizzano in modo abbastanza omogeneo le motivazioni indicate dagli intervistati:

1. TEMPO / LAVORO

per mancanza di tempo da dedicare allo studio , perché gli studi si sono rivelati incompatibili con il lavoro che svolgeva, perché ha trovato un lavoro

2. ORGANIZZAZIONE DELLA DIDATTICA E DELLA STRUTTURA

per la disorganizzazione dei corsi e delle lezioni, per la carenza di laboratori per le esercitazioni, per la scarsa disponibilità dei docenti, per la disorganizzazione degli uffici (segreterie, amministrazione), per la bassa qualità dei servizi di accoglienza offerti (mense, attività ricreative). perché è rimasto/a deluso/a dall'esperienza universitaria, perché l'organizzazione dei corsi non teneva conto delle esigenze dei non frequentanti

3. LO STUDIO IN GENERALE

perché il corso di studi si è rivelato troppo difficoltoso, perché gli studi si sono dimostrati poco rigorosi, perché si è accorto/a che il corso di studio era poco professionalizzante, perché la preparazione ricevuta nella scuola superiore si è rivelata insufficiente

4. ERRORE NELLA SCELTA

perché si è accorto/a di avere sbagliato la scelta del corso di studio, perché ha scoperto di essere interessato ad altri tipi di studio, perché sceglierà o ha già scelto un ateneo più prestigioso, perché è diminuito il suo interesse per gli studi intrapresi, perché si reso/a conto di non avere più voglia di studiare

5. MOTIVI FAMILIARI E DI COSTO DEGLI STUDI

per l'impossibilità di sostenere i costi degli studi (compresi trasporti e alloggio), perché aveva il desiderio di rendersi indipendente dalla sua famiglia, per motivi legati alla distanza da casa o alla difficoltà negli spostamenti per motivi familiari o di salute (sua o di suoi familiari), per la carenza di spazi per lo studio

Tabella 43

Distribuzione percentuale dei macro-indicatori che sintetizzano le motivazioni sottese all'abbandono

Macro-indicatore	Valore %

[24] R.N. Shepard, A.K. Romney, S.B. Nerlove, "Multidimensional Scaling. Theory and Applications in the Behavioral Sciences, Seminar Press, New York, 1972

TEMPO / LAVORO	32.56
MOTIVI FAMILIARI E DI COSTO DEGLI STUDI	20.51
ERRORE NELLA SCELTA	19.97
ORGANIZZAZIONE DELLA DIDATTICA E DELLA STRUTTURA	18.73
LO STUDIO IN GENERALE	8.23
Totale	***100.00***

Come precedentemente supposto il gruppo di motivazioni che ha la maggiore incidenza sul drop out è legato alla poca disponibilità dell'immatricolando a dedicarsi allo studio per motivi di lavoro. I successivi due "gruppi" in ordine di importanza riguardano motivazioni su cui la Facoltà ha poche possibilità di incidere se non con delle azioni di orientamento in entrata per evitare che il futuro immatricolando sbagli a scegliere il corso di laurea o la Facoltà. Le ultime due categorie riguardano, invece, degli aspetti che sono certamente legati a responsabilità dirette della nostra Facoltà e che potrebbero costituire argomento di discussione in sedi opportune al fine di identificare delle migliorie, attraverso processi di riorganizzazione e di orientamento, che permettano una seppur parziale riduzione del numero di abbandoni. A conclusione di quanto finora detto si deve aggiungere (Tab. 44) che solo il 18.4% degli intervistati si riscriverà all'università e di questi il 76.5% alla stessa Facoltà di Economia.

Graf. 1

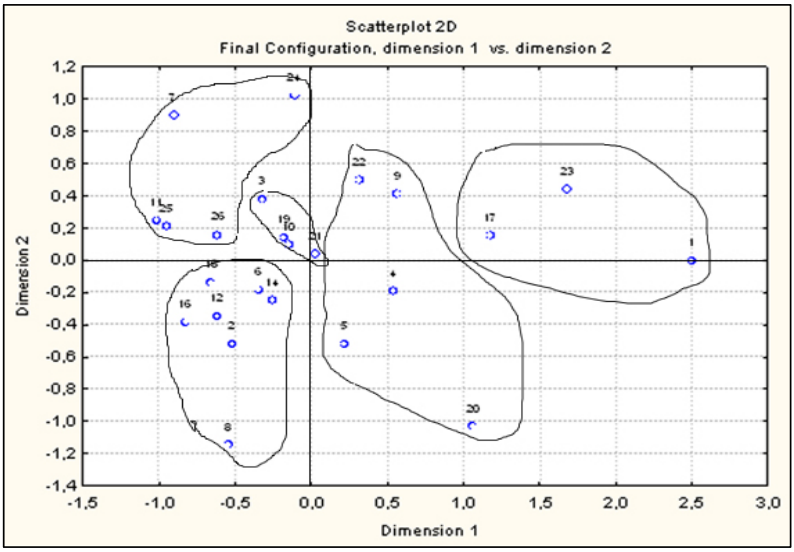

Tabella 44

Continuerà gli studi ? ***Valori %***		
NON SA		30.43
NON SI ISCRIVERÀ PIÙ ALL'UNIVERSITÀ		27.17
REGOLARIZZA ISCRIZIONE ENTRO QUEST'ANNO ACCADEMICO		23.91
di cui	*ad un altro Ateneo*	*50.0*
	ad un altro C.L. altra Facoltà Univ. Messina	*31.8*
	ad un altro corso di laurea della stessa Facoltà	*0.0*
	allo stesso corso di laurea	*18.2*
REGOLARIZZA ISCRIZIONE IN UN PERIODO SUCCESSIVO		18.48
di cui	*ad un altro Ateneo*	*11.8*
	ad un altro C.L. altra Facoltà Univ. Messina	*11.8*
	ad un altro corso di laurea della stessa Facoltà	*11.8*
	allo stesso corso di laurea	*64.7*
Totale complessivo		***100.00***

A completamento dell'indagine è stato chiesto agli intervistati di esprimere un giudizio sulla qualità di alcuni aspetti del servizio universitario. Ovviamente, come risulta dalle percentuali (Tab.45), in molti casi gli intervistati non hanno saputo esprimere un giudizio, pertanto osservando solo gli aspetti su cui si sono dichiarati più preparati ciò che traspare è che esiste un insoddisfazione per i punti di informazione e per i servizi erogati dagli uffici amministrativi.

Tabella 45

Valutazione dei Rispondenti su alcune strutture e servizi della nostra Facoltà
Valori %

	Molto soddisfatto	abbastanza soddisfatto	poco soddisfatto	per nulla soddisfatto	non so rispondere	***Totale***
Aule di lezione	9.78	36.96	25.00	10.87	17.39	***100.00***
Spazi di studio	7.61	18.48	25.00	11.96	36.96	***100.00***
Organizzazione dei corsi	7.61	36.96	25.00	14.13	16.30	***100.00***
Laboratori	3.26	19.57	8.70	9.78	58.70	***100.00***
Biblioteca	6.52	25.00	8.70	7.61	52.17	***100.00***
Sito internet	15.22	38.04	14.13	6.52	26.09	***100.00***
Servizi igienici	3.26	25.00	22.83	10.87	38.04	***100.00***
Spazi di ristoro	3.26	27.17	17.39	6.52	45.65	***100.00***
Spazi di incontro e aggregazione	3.26	23.91	19.57	7.61	45.65	***100.00***
Sicurezza garantita dall'università	6.52	33.70	11.96	7.61	40.22	***100.00***
Uffici amministrativi (segreterie, etc.)	7.61	26.09	31.52	21.74	13.04	***100.00***
Punti di informazione (bacheche, etc.)	6.52	25.00	30.43	17.39	20.65	***100.00***
Parcheggio	3.26	5.43	6.52	29.35	55.43	***100.00***
Attrezzature per disabili	3.26	6.52	4.35	6.52	79.35	***100.00***

6. Conclusioni

6.1 Lo scenario.

L'orientamento alla formazione, durante il periodo scolastico, si rivolge agli studenti delle scuole medie, superiori e dell'Università. Questa, altri Enti[25](Ordini professionali ecc.), Associazioni categoriali (di imprenditori e lavoratori) ed imprese organizzano attività di orientamento e formazione a favore di diplomati e laureati, che vogliono inserirsi nel mondo del lavoro o aggiornare le loro competenze professionali (corsi di formazione, master di I e II livello, corsi di aggiornamento, corsi di formazione lavoro ecc). Tutte queste azioni di orientamento (forse sarebbe più opportuno parlare di orientamento–formazione) hanno oggi acquisito un ruolo determinante nelle politiche pubbliche e private di finanziamento.

Costituisce anche parte essenziale delle azioni di orientamento la formazione degli orientatori, degli studenti e/o laureati, di tutto il personale che, a diverso titolo dalla docenza, partecipa a tali azioni, a favore di studenti e/o laureati. E'opportuno ricordare che nelle azioni di orientamento-formazione, successive al conseguimento dei summenzionati titoli di studio, i destinatari possono essere: soggetti in attesa di prima occupazione, lavoratori dipendenti ed autonomi, imprenditori, professionisti, persone che hanno perso il lavoro o che vogliono migliorare le loro condizioni di lavoro. Anche se l'Università ha, ed avrà sempre più, un ruolo primario in tutte le fasi e nelle più svariate azioni di orientamento-formazione, in questa sede saranno presi in esame solo alcuni degli aspetti relativi al periodo universitario ed a quello che segue il primo inserimento nel mondo del lavoro.

Problematiche sostanzialmente simili a quest'ultimo punto sono quelle riguardanti gli studenti-lavoratori che, completato il percorso universitario, desiderano migliorare la propria posizione lavorativa.

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[25] Si possono includere nelle azioni dell'orientamento scolastico anche quelle che si attuano a favore degli studenti delle scuole medie. Una trattazione a parte merita l'orientamento al lavoro fatto dagli istituti professionali e dagli istituti superiori.

E' altrettanto importante ricordare che una moderna e sistemica azione di orientamento-formazione deve essere progettata e realizzata secondo modelli capaci di verificare dinamicamente l'efficacia del servizio reso, i livelli di soddisfazione dei destinatari (laureati, lavoratori, ecc.) e quelli degli eventuali "utilizzatori" (Enti pubblici e privati, imprese, ecc.).
Il riferimento alle politiche di investimento da parte di Enti pubblici e privati collegate alle azioni di orientamento-formazione, nasce anche, dalla constatazione che i criteri, le modalità, le priorità nell'erogazione dei finanziamenti - sia ordinari che straordinari – sono esplicitamente mirati a premiare la produttività e l'efficacia gestionale. Ciò presuppone una capacità operativa, caratterizzata da una sistemica razionalizzazione e modernizzazione dei processi e da una ricerca continua di miglioramento dei servizi resi.
Questa azione di stimolo all'ammodernamento, al rinnovamento del sistema universitario rappresenta comunque il mezzo e non il fine. Infatti, l'aspetto di gran lunga più importante circa il ruolo e le finalità della formazione è certamente quello legato alla valenza strategica del capitale umano che da essa sostanzialmente dipende.
La risorsa principale di un territorio e della comunità che in esso risiede si sostanzia, oggi più che mai, in un'ampia e diffusa presenza di cittadini culturalmente consapevoli e convinti interpreti di un sistema di elevati valori etici e morali.
E' questa grande "risorsa" infatti la premessa, il presupposto indispensabile per formare donne e uomini preparati ad affrontare le grandi sfide dell'avanzamento rapido delle conoscenze, delle competenze ed abilità, delle professionalità e quindi capaci di competere sul piano sociale ed economico, in un sistema planetario globalizzato.
Esaminato lo scenario, in cui si colloca questo lavoro, che ha come obiettivo specifico l'orientamento svolto dall'Università, passiamo all'esame dei problemi di orientamento della Facoltà di Economia dell'Università di Messina.
Va sottolineato, comunque, che la missione più alta dell'Università, prevista dalla nostra Costituzione, è quella di garantire, in condizioni e spirito di autonomia e libertà, l'attività congiunta della ricerca scientifica e dall'alta formazione. L'Università deve, comunque, operare per:

- razionalizzare, controllare e meglio finalizzare la spesa per raggiungere un'alta produttività;
- misurare costantemente e perseguire alti livelli di efficacia, adeguandosi alle mutevoli esigenze del territorio;
- attenzionare, a livello tattico e strategico, i sistemi di valutazione dell'ANVUR (Agenzia Nazionale di Valutazione del Sistema Universitario e della Ricerca) [26].

Con riferimento ai suddetti obiettivi, questo lavoro si prefigge di meglio comprendere i fenomeni di ritardo nel processo formativo, di dispersione, in definitiva, di inefficienza della facoltà di Economia dell'Università di Messina.

6.2 "Drop out"

I punti 3 e 4 (rispettivamente, "Drop out" ed il "fenomeno del Drop out in Italia") hanno consentito di mettere in evidenza – con il supporto di un'ampia ed aggiornata bibliografia – una serie di "motivazioni/cause" che certamente concorrono a determinare alti livelli di "dispersione". Trovandomi in totale accordo con quanto scritto dal prof. Ricca riguardo al punto 4, è possibile affermare che le motivazioni dell'alto tasso di dispersione (abbandono) pari al 60% non possono, giustamente, trovare spiegazione nei " titoli di studio dei genitori" e nel " tipo di scuola secondaria frequentata". Solo un esame "disaggregato" e più attento, che prende in considerazione l'età, l'eventuale occupazione, e più in generale le classi di categorie citate nel lavoro di Fasanella e Tanucci, può consentire una lettura migliore dei dati OCSE.[27] Altrettanto importante è osservare, come peraltro è stato opportunamente messo in evidenza nel punto 5.2, l'incidenza degli abbandoni per Facoltà. Da una attenta analisi, infatti, appare evidente una stretta connessione e, conseguentemente, una significativa diversità di dati tra le varie Facoltà. Dai risultati (in valori assoluti e percentuali) riportati in Tab.2, relativi all'Ateneo messinese, appare evidente che le percentuali di abbandono si riducono notevolmente nei seguenti casi:

- quando la scelta del corso di studio è fortemente motivata ed adeguatamente maturata sul piano dell'impegno (sacrificio personale) necessario;

[26] Come già detto nella premessa del lavoro in questa fase le azioni di valutazione e di indirizzo per la valutazione sono affidate dal Governo al CNUSU ed al CIV.
[27] Vedesi testo sulle classi categoriali riportate nel paragrafo 3 e relativa nota 9.

- quando è stata valutata la necessità di possedere un bagaglio di "saperi minimi per affrontare le verifiche dei primi anni o si è provveduto a compensare gli eventuali "debiti formativi", (abbastanza facilmente rilevabili);
- in presenza di situazioni familiari e/o ambientali favorevoli;
- quando si è in condizione di studiare a tempo pieno (obbligo di frequenza formalizzato od anche sostanzialmente applicato per consuetudine).

Nonostante in tutte le Facoltà esistano giovani che hanno le caratteristiche sopramenzionate, sussistono significative differenze in relazione ai vari corsi di studio ed, in certa qual misura, al contesto socio-economico in cui i vari atenei operano. Infatti sono evidenti, anche se non generalizzabili, le differenze percentuali tra Facoltà come Medicina, Veterinaria o Farmacia e Scienze della Formazione o Giurisprudenza.

Il prof. Ricca al punto 5.2 sottolinea (con il supporto di una indagine, svolta nell'Ateneo di Milano – Bicocca, nota 25) la presenza, numericamente significativa, di "una nuova categoria di studenti che si iscrivono ad un corso di primo livello, attratti, ……, dalla brevità delle lauree triennali, pur non avendo tempo da dedicare allo studio ma avendo la possibilità di mettere in pausa il percorso universitario ogni qualvolta lo desiderano…….".

Da questa importante considerazione scaturisce la necessità di una sollecita attuazione, da parte della nostra Facoltà, delle convenzioni per studenti lavoratori; esse dovranno indicare un periodo fisiologico di formazione superiore ai 3 anni (ad esempio 5 o 6) ed una corrispondente dilazione (in 5 o 6 anni) del pagamento delle tasse universitarie. A queste convenzioni dovranno essere collegati moduli didattici specifici, più rispondenti alle esigenze degli studenti lavoratori, o con problematiche simili a questi ultimi.[28]

Il prof. Ricca evidenzia inoltre, nel punto 5.3, la negativa riduzione di iscritti provenienti da altre province e soprattutto da altre regioni (nella valutazione CNVSU quest'ultimo aspetto, è fortemente penalizzante per la nostra Facoltà e per lo stesso Ateneo). Le cause di tale andamento trovano spiegazione nella crescente competitività degli Atenei operanti nel nostro bacino di utenza. Questo fenomeno può essere però ostacolato, ed in parte compensato innovando i processi di formazione ed i servizi resi, che potranno essere adeguati alle esigenze di particolari segmenti di fruitori. Fortemente connesso a questi aspetti trattati, è il fenomeno dell'alta presenza di studenti fuori corso (tabelle dall'11 al 19). Anche per quanto concerne il passaggio degli iscritti dal primo al secondo anno (Tab. 27 e 28), è importante quanto osservato dal prof. Ricca relativamente all'alta percentuale di "matricole" che non rinnovano l'iscrizione dopo il 1° anno e di studenti che si immatricolano in età avanzata (studenti lavoratori o che hanno svolto una qualsiasi attività di lavoro). Dai dati riportati nella Tab. 34 e dalla utilizzazione dell'analisi di regressione logistica (nota 29), appare evidente una propensione all'abbandono "37 volte maggiore" da parte di studenti che non hanno superato alcun esame rispetto a chi, invece, ne ha superato anche solamente uno. Allo stesso modo l'analisi dei dati riportati nella tabella 35 ha consentito all'autore di rilevare un più alto rischio di abbandono per le donne e per i residenti in regioni diverse dalla Sicilia e dalla Calabria (i dati si riferiscono ad immatricolati nel 2005-06 che non hanno rinnovato l'iscrizione nel 2007). Anche ciò incide negativamente nella valutazione CNVSU e presuppone l'attuazione tempestiva di azioni rivolte alla modifica di tale andamento.

[28] Le Facoltà di Economia hanno tradizionalmente un alto numero di studenti lavoratori; in particolare in quelle meridionali la percentuale di questi, rispetto agli studenti a tempo pieno, risulta ancora più elevata. Dall'analisi dei dati citati e rilevati direttamente dal Prof. Bruno Ricca (punto 5.4 "Le motivazioni dell'abbandono: un'indagine campionaria"), appare evidente che le principali cause di degli abbandonoisono: "la mancanza di tempo da dedicare allo studio" (sostanzialmente la condizione di studenti lavoratori), "i motivi familiari" e "la distanza da casa."

L'analisi dei dati Alma Laurea sul "Profilo dei laureati 2006" mette in evidenza che il 50,9% degli studenti della nostra Facoltà ha oltre 27 anni rispetto al 25% delle facoltà di Economia del resto d'Italia.

Inoltre la percentuale degli studenti in regola (in corso) dell'Università di Messina, pari all'11,9%, risulta inferiore rispetto alla media nazionale pari al 32,9%.

E' possibile notare che la percentuale degli studenti iscritti al 5° anno fuori corso è notevolmente superiore a Messina (39,3%) rispetto la media nazionale (14,1%).

Nel punto 5 "Condizioni di studio" vengono analizzati i dati relativi alla frequenza delle lezioni da parte degli studenti; dall'analisi dei risultati appare evidente che la percentuale dei discenti frequentanti il 75% delle ore di lezione della Facoltà di Economia dell'Università di Messina è notevolmente inferiore (29,2%) rispetto alla media nazionale (62,6%).

Gli altri dati su questo punto confermano una scarsa frequenza che si coniuga sia con l'alta percentuale di studenti lavoratori che con l'elevata età media.

Dall'analisi dei dati riportati nel punto 6 "Lavoro durante gli studi" è possibile osservare che la percentuale di studenti lavoratori per la Facoltà di Economia dell'Università di Messina è notevolmente alta e si aggira intorno al 72,9%.

Tutti questi elementi trovano ulteriori conferme negative per gli studenti residenti in altre regioni e/o che hanno altri impedimenti – come spesso accade per alcune donne che hanno per vari giustificati motivi meno tempo a disposizione da dedicare allo studio -.

6.3 Considerazioni di sintesi e possibili azioni conseguenti.

Riassumendo, tra i principali punti di debolezza[29] della nostra Facoltà si evidenziano:

a. un andamento crescente del numero di immatricolati che non proseguano il percorso universitario;
b. un trend decrescente del numero di esami (e relativi crediti formativi universitari) mediamente conseguiti negli anni di corso e nel primo anno fuori corso.

Più specificamente si osserva:

- un trend crescente del numero di abbandoni da parte di studenti lavoratori;
- un trend decrescente del numero di immatricolati provenienti da altre regioni, ad eccezione del Corso di laurea in ETA.

La distanza della Facoltà dal luogo di residenza può rappresentare un altro punto di debolezza, soprattutto se aggravato da problemi familiari e/o personali che limitano la disponibilità di tempo libero dello studente.

Vi è chiaramente una stretta connessione tra la mancanza di tempo da dedicare allo studio (lavoro, impegni familiari [maggiori per le donne], distanza della Facoltà dal luogo di residenza) e la difficoltà a superare esami, con conseguente ritardo nella carriera scolastica o anche l'abbandono della stessa.

Un piano quindi mirante a "prevenire" o comunque "contenere"[30] ritardi ed abbandoni dovrebbe:

1 - <u>progettare ed attuare convenzioni e moduli didattici specifici a favore degli studenti lavoratori</u> e di quelli che, a causa di problemi familiari e/o personali (in particolare donne), hanno un minore tempo a disposizione per frequentare le lezioni e per studiare;

1.1 - <u>progettare ed attuare, per residenti in altre regioni, convenzioni e moduli didattici specifici decentrati</u>.

La nostra Facoltà ha esperienza consolidata in corsi per lavoratori, corsi serali e corsi "full immersion"; tali esperienze sono precedenti preziosi e devono servire da stimolo per realizzare, in maniera sistematica, specifici progetti didattici, che prevedano particolari diari di lezioni, esercitazioni, seminari, verifiche ecc.

I docenti, che registrano andamenti "anomali"[31], sul piano del rendimento da parte degli studenti, dovrebbero essere protagonisti di innovative azioni per invertire tale tendenza. In questo senso la didattica per piccoli gruppi ed alcune mirate azioni di tutorato (inseguimento a distanza ecc.) potrebbero costituire una efficace prevenzione agli alti tassi di abbandono. Quindi bisognerebbe motivare gli studenti svantaggiati a frequentare le lezioni, stimolarli al dialogo diretto con i docenti, incrementando il loro impegno e migliorandone le capacità di apprendimento al fine di superare le verifiche (esami), e di conseguire i relativi crediti formativi.

In questo senso sarebbe opportuno eseguire un monitoraggio attento dei corsi, delle aree disciplinari, delle singole discipline per evidenziare andamenti, con "picchi" troppo alti o troppi bassi sul piano del rendimento. In questi casi la ricerca delle cause, con l'aiuto dei docenti direttamente interessati, potrebbe stimolare gli studenti più volenterosi a meglio focalizzare il loro impegno ed innalzare conseguentemente l'efficacia complessiva dell'azione didattica.

Per quanto riguarda il punto 1.1, vale quanto detto al precedente punto 1, puntando anche sul decentramento territoriale. In tal senso l'esperienza biennale di Fiuggi ha dimostrato come si possa trovare una risposta efficace ai problemi degli studenti disagiati con limitato tempo a disposizione (studenti lavoratori, ecc.), ancorché residenti in altra regione. L'avere realizzato una iniziativa sperimentale nell'agosto- settembre 2006 (anno accademico 2005-06), l'averla consolidata nello stesso periodo del 2007 (anno accademico 2007-08) con un alto successo di adesioni (iscrizioni) dalla regione Lazio, ha dimostrato che concentrare, in periodi relativamente brevi ma intensi (full immersion), il "dialogo" tra docenti e discenti, fortemente motivati, può dare risultati eccezionali. È opinione largamente condivisa da professori e studenti che hanno vissuto questa innovativa esperienza, che il corso estivo di ETA (Economia del Turismo e dell'Ambiente), coniugato con la forte vocazione turistico-ambientale della provincia di Frosinone, ha offerto agli studenti con difficoltà di normale frequenza la possibilità di godere di una interessante ed utile opportunità.

Questa esperienza di decentramento didattico, programmata nei mesi estivi per consentire l'utilizzazione del periodo feriale, o comunque progettata per rispondere a logiche richieste,

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[29] Si tratta soprattutto di elementi che hanno un impatto negativo su criteri di valutazione degli Atenei da parte del CNVSU.

[30] Nel punto 3 del testo, opportunamente, si riporta la distinzione tra azioni di orientamento destinate al "recupero" o "salvataggio" di studenti in ritardo e quelle, invece, rivolte alla "prevenzione" e/o "contenimento" della dispersione (abbandono).

[31] Puntando solo sul diretto e convinto impegno dei docenti che registrano alte percentuali di insuccessi nelle verifiche, si possono progettare e realizzare strategie vincenti per migliorare l'efficacia complessiva del processo formativo dei singoli corsi di laurea.

motivate ed adeguatamente supportate, può costituire nel rispetto della dovuta progressività, e con le opportune verifiche periodiche, una scelta nuova per migliorare la “student retention” della nostra Facoltà. In definitiva l’obiettivo primario è quello di trasformare alcuni “vincoli” tradizionali della Facoltà (alto numero di studenti lavoratori, età media più alta ed alta percentuale di studenti provenienti da altra regione) in ottime “opportunità”. Queste ultime, se coniugate con una sempre più alta politica di miglioramento dei servizi agli studenti - in sinergico confronto con la realtà socio-economica ed in un organico sistema di relazioni internazionali -potrebbero rafforzare le capacità di attrazione di iscritti da altre regioni[32] e contribuire, in maniera significativa, al miglioramento della valutazione complessiva dell’Ateneo. Tali importanti iniziative potrebbero costituire anche un’ utile premessa per progettare piani di recupero (salvataggio). Sarà inoltre, possibile, con l’aiuto di piccoli adeguamenti (ad esempio progetti personalizzati), trovare risposte efficaci al “recupero” di studenti in notevole ritardo, ma disposti ad un forte impegno per non abbandonare l’Università. In definitiva, esistono nella Facoltà di Economia dell’Università di Messina le condizioni soggettive (esperienza personale dei docenti) ed oggettive (conoscenza delle cause ed esperienze concrete di innovazione di processo e di prodotto) per invertire un andamento non positivo dipendente dalla istituzione di nuove regole per la valutazione del funzionamento dei corsi di laurea. E’essenzialmente compito di noi docenti, degli amministrativi e dei tecnici, adeguarsi, nei tempi più rapidi possibili, alle mutate esigenze, stabilite dalla nuova normativa, superando vecchi steccati e sterili convincimenti di autorefertenzialità

[32] Aspetto non secondario, ampliamente condiviso tra i docenti che hanno operato nel corso estivo di Fiuggi, è quello di essersi creato un proficuo e stimolante clima di collaborazione tra discenti e docenti. Gli uni e gli altri, vivendo in un laborioso ed intenso clima comunitario, si sentono spinti a profondere il migliore impegno per non deludere le reciproche attese. Si crea un ciclo virtuoso, riscontrabile solitamente nelle attività seminariali intense o in quelle occasioni di formazione e di ricerca nelle quali in un clima di full immersion si riesce ad mettere in campo le migliori energie sia dei docenti che dei discenti.
Tale clima di operoso impegno è stato utile anche a studenti non residenti nel Lazio che hanno voluto utilizzare questa opportunità.

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TORIC IDEALS AND CYCLES

GIANCARLO RINALDO

ABSTRACT. Let $\mathcal{G}$ be a simple undirected graph and $\mathcal{B}$ a bipartite graph associated to $\mathcal{G}$, we define the monomial subring $K[\mathcal{B}]$ and its toric ideal $P(\mathcal{B})$. A minimal set of generators of $P(\mathcal{B})$ describe all the cycle of the graph $\mathcal{G}$. We give an implementation in CoCoA to calculate all the cycles of $\mathcal{G}$.

INTRODUCTION

Let $\mathcal{G}$ be a simple graph on the vertex set $V(\mathcal{G}) = \{x_1, ..., x_n\}$ and edge set $E(\mathcal{G}) = \{x_i x_j : x_i \text{ adjacent to } x_j\}$, $q = |E(\mathcal{G})|$ the number of edges.

We define a new graph $\mathcal{B}$ starting from $\mathcal{G}$ such that

(1) $V(\mathcal{B}) = \{x_1, \ldots, x_n\} \cup \{x_{ij} | x_i x_j \in E(\mathcal{G})\}$;
(2) $E(\mathcal{B}) = \{x_i x_{ij}, x_j x_{ij} | \forall x_i x_j \in E(\mathcal{G})\}$.

It is easy to observe that $\mathcal{B}$ is bipartite with $n + q$ vertices and $2q$ edges.

We consider the epimorphism Φ

$$\Phi : S = K[T_{ij}, T_{ji} | \{x_i, x_j\} \in E(\mathcal{G})] \longrightarrow K[\mathcal{B}],$$

its kernel is the graded toric ideal generated by graded binomial, that we call $P(\mathcal{B})$.

In Theorem 1.3 (see [5]) we show that there exists a bijection between a minimal set of generators of $P(\mathcal{B})$ and the set of the cycles of $\mathcal{G}$. Moreover, each of these generators of $P(\mathcal{B})$ describe completely a cycle $C \in \mathcal{G}$, its length, its vertices, its edges. Therefore using Theorem 1.3 and Algorithm 1.8 (see [5]), we have a nice algebraic method to calculate all the cycles of $\mathcal{G}$.

In Section 1 we give an implementation of Algorithm 1.8 in CoCoA (see [3]), a computer algebra system.

By this implementation the author has been able to reach interesting conjectures on the algebraic properties of $K[\mathcal{B}]$ (Section 2).

In particular Krull dimension, projective dimension and Castelnuovo-Mumford regularity of $K[\mathcal{B}]$, seems to be strictly related to combinatorial properties of the graph $\mathcal{G}$.

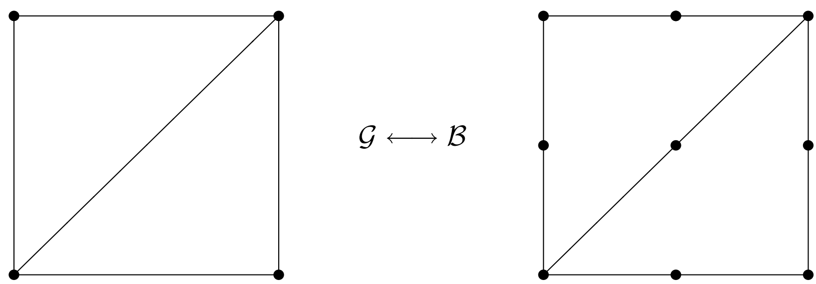
1. Toric ideals and Graphs

Let $\mathcal{G}$ be a simple undirected graph with n vertices, $V(\mathcal{G}) = \{x_1, \ldots, x_n\}$, and q edges, $E(\mathcal{G}) = \{x_i x_j | x_i \text{ adjacent to } x_j\}$

Definition 1.1. *Let $\mathcal{B}$ a graph related to $\mathcal{G}$ such that*

(1) $V(\mathcal{B}) = \{x_1, \ldots, x_n\} \cup \{x_{ij} | x_i x_j \in E(\mathcal{G})\}$;
(2) $E(\mathcal{B}) = \{x_i x_{ij}, x_j x_{ij} | \forall x_i x_j \in E(\mathcal{G})\}$.

It is easy to observe that $\mathcal{B}$ is bipartite, since every cycle is even (Koenig Theorem, [4]), with $n + q$ vertices and $2q$ edges (see Figure).



Let R be the polynomial ring $R = K[x_i, x_{ij} \in V(\mathcal{B})]$ where K is a field, with the standard grading. A monomial $f_{ij} = x_i x_{ij}$ is said a generator of $\mathcal{B}$. Let f_{ij}, f_{ji}, $\forall x_i x_j \in E(\mathcal{G})$ be the generators of $\mathcal{B}$, the monomial subring $K[\mathcal{B}] \subset R$ related to the graph $\mathcal{B}$ is

$$K[\mathcal{B}] = K[f_{ij}, f_{ji} | x_i x_j \in E(\mathcal{G})].$$

There exists a graded epimorphism of K-algebras

$$\Phi : S = K[T_{ij}, T_{ji} | \{x_i, x_j\} \in E(\mathcal{G})] \longrightarrow K[\mathcal{B}]$$

with $\Phi(T_{ij}) = f_{ij}$. Its kernel $P(\mathcal{B})$ is a graded prime ideal, generated by pure binomials (that is a difference between two monomials).

Remark 1.2. *If C is a cycle of $\mathcal{G}$, $C = \{x_{i_1}, x_{i_2}, \ldots, x_{i_r}\}$ then there exists a binomial b_c in P such that $b_c = T_{i_1 i_2} T_{i_2 i_3} \cdots T_{i_r i_1} - T_{i_2 i_1} T_{i_3 i_2} \cdots T_{i_1 i_r}$. In fact $\Phi(b_c) = f_{i_1 i_2} f_{i_2 i_3} \cdots f_{i_r i_1} - f_{i_2 i_1} f_{i_3 i_2} \cdots f_{i_1 i_r} = 0$.*

It is easy to observe that the sequence of vertices $\{i_1, i_2, \ldots, i_r\}$ describe the cycle C in one direction say the directed cycle C^+, while the sequence of vertices $\{i_r, i_{r-1}, \ldots, i_1\}$ describe the cycle C in the other direction say the directed cycle C^-, therefore we write each binomial b_c as $T_{c^+} - T_{c^-}$.

Theorem 1.3 (see [5]). *Let G_P the set of binomial*

$$G_P = \{b_c | C \text{ is a cycle of } \mathcal{G}\}.$$

Then G_P is a minimal set of generators of the toric ideal P.

We recall the following

Definition 1.4. *A* cutpoint *of a graph is a vertex whose removal increases the number of (connected) components.*

A block *of a graph is a maximal subgraph without cutpoints.*

Corollary 1.5. *Let $\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_r$ be the blocks of $\mathcal{G}$. Then*

$$K[\mathcal{B}] \cong \otimes_{i=1}^{r} K[\mathcal{B}_i]$$

where $\mathcal{B}_i$ is the bipartite graph related to the block $\mathcal{G}_i$ for $i = 1, \dots, r$.

Proof. We observe that if v is a cutpoint between two blocks $\mathcal{G}_1$ and $\mathcal{G}_2$ then there is no cycles passing through v and connecting a vertex of $\mathcal{G}_1$ with a vertex of $\mathcal{G}_2$. Therefore by Theorem 1.3 we have the assertion. □

Example 1.6. *Let $\mathcal{G}$ be a triangle with vertices $V(\mathcal{G}) = \{x_1, x_2, x_3, x_4, x_5\}$ and edges $E(\mathcal{G}) = \{\{x_1, x_2\}, \{x_1, x_3\}, \{x_2, x_3\}, \{x_1, x_4\}, \{x_1, x_5\}, \{x_4, x_5\}\}$. That is $\mathcal{G}$ is composed of two triangles having the vertex x_1 in common. In this case we have exactly two blocks that are the two triangles.*

Let $\mathcal{B}$ the bipartite graph related to $\mathcal{G}$ and $\mathcal{B}_1$ and $\mathcal{B}_2$ the two bipartite graphs related to the triangles. If we consider the mapping

$$T_{12} \to x_1x_{12}, T_{21} \to x_2x_{12}, T_{13} \to x_1x_{13}, T_{31} \to x_3x_{13}, T_{23} \to x_2x_{23},$$
$$T_{32} \to x_3x_{23}, T_{14} \to x_1x_{14}, T_{41} \to x_4x_{14}, T_{15} \to x_1x_{15}, T_{51} \to x_5x_{15},$$
$$T_{45} \to x_4x_{45}, T_{54} \to x_5x_{45}$$

then

$$P(\mathcal{B}) = (T_{12}T_{23}T_{31} - T_{13}T_{32}T_{21}, T_{14}T_{45}T_{51} - T_{15}T_{54}T_{41}) = P(\mathcal{B}_1) + P(\mathcal{B}_2).$$

That is

$$K[\mathcal{B}] \cong K[T_{12}, T_{21}, T_{13}, T_{31}, T_{23}, T_{32}, T_{14}, T_{41}, T_{15}, T_{51}, T_{45}, T_{54}]/P(\mathcal{B})$$

and $K[\mathcal{B}]$ is isomorphic to

$$K[T_{12}, T_{21}, T_{13}, T_{31}, T_{23}, T_{32}]/P(\mathcal{B}_1) \otimes K[T_{14}, T_{41}, T_{15}, T_{51}, T_{45}, T_{54}]/P(\mathcal{B}_2).$$

Remark 1.7. *By Remark 1.2 and Theorem 1.3 follows that the initial monomials of the generators of $P(\mathcal{B})$ are sufficient for the description of the cycles of $\mathcal{G}$.*

Algorithm 1.8 (see [5]). *Input: A graph $\mathcal{G}$; Output: Cycles of $\mathcal{G}$.*

(1) *Define the incidence matrix of the graph $\mathcal{B}$ induced from $\mathcal{G}$;*
(2) *Find the lattice basis of* $\ker \varphi$ *that is cycle space of $\mathcal{B}$;*
(3) *Define the ring $S = K[T_{i,j}, T_{j,i} | \{x_i, x_j\} \in E(\mathcal{G})]$;*

(4) *Define* $P_0 = (T^{\mathbf{u}} - T^{\mathbf{v}} | \mathbf{u} - \mathbf{v} \in \ker \varphi) \subset S$;
(5) *For each* $i < j$,$x_i x_j \in E(\mathcal{G})$ *do* $P_{k+1} = P_k : T_{i,j}^{\infty}$, *increment* k;
(6) *Let* $C = \mathrm{in}(P_k)$, *for each generator of* C *print the corresponding cycle of* $\mathcal{G}$.

Here is a translation in CoCoA, with an example of computation of the cycles of K_4.

```
 -- Input: Definition of K4
L:=[[1,2],[1,3],[1,4],[2,3],[2,4],[3,4]];
Nl:=Len(L);
Nv:=Max(ConcatLists(L));

-- Step 1) Calculation of the incidence matrix of the graph B.
MInc:=NewMat(Nv+Nl, 2*Nl,0);

For I:=1 To Nl Do
  MInc[L[I,1], 2*I-1]:=1;
  MInc[L[I,2], 2*I]:=1;
  MInc[Nv+I,2*I-1]:=1;
  MInc[Nv+I,2*I]:=1;
EndFor;

-- Step 2) Calculation of the lattice basis of the kernel.
MCyc:=LinKer(MInc);

-- Step 3) Definition of the ring
NFunCyc:=Len(MCyc);
Use S::=Q[t[1..2*Nl]];
B:=NewList(NFunCyc);

-- Step 4) Translation of the lattice basis into binomials.
For I:=1 To NFunCyc Do
  Mon1:=1;
  Mon2:=1;
  For J:=1 To 2*Nl Do
    If MCyc[I,J]=1 Then Mon1:=Mon1*t[J]
    Elsif MCyc[I,J]=-1 Then Mon2:=Mon2*t[J]
    EndIf;
  EndFor;
  B[I]:=Mon1-Mon2;
EndFor;

-- Step 5) Saturation with respect to the indeterminates
```

```
-- (Internal command of CoCoA).
P:=Toric(B);

-- Step 6) For each binomial print the corresponding cycle.
Cycles:=Gens(LT(P));
NCyc:=NewList(Nv,0);
For I:=1 To Len(Cycles) Do
  Exp:=Log(Cycles[I]);
  Cycle:=[];
  For J:=1 To 2*Nl Do
    If Exp[J]=1 Then
      If Mod(J,2)=1 Then
           Append(Cycle,[L[Div(J+1,2),1], L[Div(J+1,2),2]])
      Else
           Append(Cycle,[L[Div(J,2),2], L[Div(J,2),1]]);
      EndIf;
    EndIf;
  EndFor;
  NCyc[Len(Cycle)]:=NCyc[Len(Cycle)]+1;
  VStart:=Min(ConcatLists(Cycle));
  V:=VStart;
  Repeat
    Print(V,".");
    I:=0;
    Repeat
      I:=I+1;
    Until Cycle[I,1]=V;
  V:=Cycle[I,2];
  Until V=VStart;
  PrintLn;
EndFor;
-- Output: Cycles of K4
1.2.4.
1.2.3.
1.3.4.
2.3.4.
1.3.2.4.
1.3.4.2.
1.2.3.4.
```

2. Invariants of $K[\mathcal{B}]$

By the implementation in CoCoA of the Algorithm 1.8 we have a good way to calculate algebraic invariants of

$$K[\mathcal{B}] = K[f_{ij}, f_{ji} | x_i x_j \in E(\mathcal{G})],$$

and $P(\mathcal{B})$, the kernel of Φ,

$$\Phi : S = K[T_{ij}, T_{ji} | \{x_i, x_j\} \in E(\mathcal{G})] \longrightarrow K[\mathcal{B}], \tag{2.1}$$

that we expect are strictly related to combinatorial properties of the graph $\mathcal{G}$.

Remark 2.1. *In our study we consider the graph $\mathcal{G}$ as a single block. In fact in Corollary 1.5 we show that if $\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_r$ are the blocks of $\mathcal{G}$, we obtain $K[\mathcal{B}] \cong \otimes_{i=1}^r K[\mathcal{B}_i]$ where $\mathcal{B}_i$ is the bipartite graph related to the block $\mathcal{G}_i$ for $i = 1, \dots, r$. Therefore studying the invariants of each single block and tensorizing the algebra obtained we find the invariants of the full algebra.*

From now on $\mathcal{B}$ is a bipartite block.

Proposition 2.2. $\dim K[\mathcal{B}] = \dim S/\operatorname{in} P(\mathcal{B}) = n + q - 1$.

Proof. Since $\mathcal{B}$ is bipartite with $n+q$ vertices and applying Corollary 8.2.13 of [8], the assertion follows. By Macaulay Theorem (see [1], Corollary 4.2.4) we have $\dim K[\mathcal{B}] = \dim S/\operatorname{in} P(\mathcal{B})$. $\square$

Remark 2.3. *In our examples we observed that many algebraic invariants seems to be equal for $P(\mathcal{B})$ and $\operatorname{in}(P(\mathcal{B}))$.*

This is supported by the fact that in some sense the second monomial of each binomial of the set of generators of $P(\mathcal{B})$ is "useless" since it represents the same cycle of $\mathcal{G}$.

We consider invariants related to a minimal free resolution of a graded module M:

(2.2)

$$\mathbb{F}: \ 0 \to \bigoplus_{i=1}^{b_g} S(-d_{g_i}) \xrightarrow{\varphi_g} \cdots \to \bigoplus_{i=1}^{b_k} S(-d_{k_i}) \xrightarrow{\varphi_k} \cdots \bigoplus_{i=1}^{b_0} S(-d_{0_i}) \xrightarrow{\varphi_0} M \to 0$$

Definition 2.4. *The length of the minimal free resolution, g in (2.2), of a graded module M is called the projective dimension of M, $\operatorname{pd}(M)$.*

Definition 2.5. *The number of generators of the cycle basis of $\mathcal{G}$ is $m(\mathcal{G}) = q - n + 1$ (see [4], Corollary 4.5.a).*

We have the following

Proposition 2.6. *$K[\mathcal{B}]$ is Cohen-Macaulay.*

Proof. Since $\mathcal{B}$ is bipartite by Theorem 5.9 of [6] $K[\mathcal{B}]$ is normal and by Theorem 6.3.5 of [1] $K[\mathcal{B}]$ is CM. □

Conjecture 2.7. *$S/\operatorname{in}(P(\mathcal{B}))$ is Cohen-Macaulay.*

In particular if Conjecture 2.7 is true, we have the following

Corollary 2.8.

$$\operatorname{pd} K[\mathcal{B}] = \operatorname{pd}(S/\operatorname{in}(P(\mathcal{B})) = m(\mathcal{G}) = q - n + 1.$$

Proof. By Auslander-Buchsbaum Theorem (see [1], Theorem 1.3.3)

$$\operatorname{pd} K[\mathcal{B}] + \operatorname{depth} K[\mathcal{B}] = \dim S,$$

where S is described in the map 2.1.

By Proposition 2.6 $K[\mathcal{B}]$ is Cohen-Macaulay therefore $\operatorname{depth} K[\mathcal{B}] = \dim K[\mathcal{B}]$ and by Proposition 2.2 we obtain

$$\operatorname{pd} K[\mathcal{B}] = \dim S - \dim K[\mathcal{B}] = 2q - (n + q - 1) = q - n + 1.$$

By Conjecture 2.7 and applying the previous arguments the assertion follows. □

Remark 2.9. *We observe that in general it is true only that $\operatorname{pd} S/ini(J) \geq \operatorname{pd} S/J$, where S is a polynomial ring on a field K and J an ideal of S.*

Definition 2.10. *The Castelnuovo-Mumford regularity of a graded module M is given by*

$$\operatorname{reg}(M) := \max(d_{k_i} - k \mid k > 0, i = 1, \ldots, b_k).$$

Definition 2.11. *The circumference of a graph $\mathcal{G}$, $c(\mathcal{G})$, is the length of any longest cycle.*

We have the following

Conjecture 2.12.

$$\operatorname{reg}(P(\mathcal{B})) = \operatorname{reg}(\operatorname{in}(P(\mathcal{B})) = c(\mathcal{G}).$$

Remark 2.13. *We observe that in general it is true only that $\operatorname{reg}/ini(J) \geq \operatorname{reg} J \geq d$, where J is an ideal of a polynomial ring on a field K, d is a the maximum degree between a minimal set of generators of J. In our case it is easy to observe that $d = c(\mathcal{G})$.*

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LONG-TERM MODELLING OF NONLINEAR HIGH CRESTS IN NARROW-BAND OCEAN WAVES UP TO THE STOKES' FIFTH-ORDER THEORY

ALESSANDRA ROMOLO

Abstract. *In this paper, the solution for the return period of a sea storm in which the highest crest exceeds a fixed threshold is considered, by taking into account effects up to the fifth order in a Stokes expansion. By analysing the nonlinear contributions to the long-term modelling of crest heights during a sea storm, it is found that higher order corrections, up to fifth-order, to the linear distributions give crest height slightly smaller than those up to second-order. This result is of interest in engineering applications for the long-term modelling of extreme crest heights.*

1. Introduction

The present paper deals with the investigation of extreme nonlinear crest heights during severe storms. This topic concerns both the statistical properties of waves in a sea state (*short-term* wave statistics) and the distribution of significant wave heights (*long-term* wave statistics).

As regards to the *short-term* statistics, the free surface displacement to the first order in a Stokes' expansion, is a stationary Gaussian random process of time. In particular, for a narrow band spectrum, Longuet-Higgins [1] proved that the crest heights are distributed by the Rayleigh's law. Boccotti ([2], [3], [4]), as corollary of his Quasi-Determinism theory, derived that to the first order, for finite bandwidth of the spectrum, the highest crests follow asymptotically the Rayleigh's law.

The non-linearity gets the crest higher and sharper and the trough lower and flatter with respect to the linear water surface process. Therefore, the probability density function of the free surface tends to deviate from the Gaussian distribution: the larger crest heights are underestimated and the deeper trough amplitudes are overestimated by the Rayleigh's distribution. This non-linear effect was firstly investigated by Longuet-Higgins [5], who proposed an exact form for the distribution of nonlinear waves up to the second-order. A narrow-band model has been derived for the second-order crest statistics (as well as for the nonlinear trough statistics) by Tayfun [6] and by Arena & Fedele [7].

More recently, the second-order effects for the crest and trough height distribution of random sea waves in an undisturbed field, for finite spectral bandwidth, were investigated by Forristall [8], Prevosto et al. [9-11], Al-Humound et al. [10], Fedele & Arena [12]. Their results, which were validated by both field data and sea wave numerical simulations, turn out to be in good agreement to each other.

Higher order effects were investigated by Dawson [13], who derived an analytical solution for narrow-band nonlinear crest distribution. The proposed solution, which was validated by means of laboratory experiments, is exact to the fifth-order in a Stokes' expansion and is valid for the deep water condition.

In this paper, the solution for the nonlinear return period $R(C)$ of a sea storm in which the maximum expected crest height exceeds a fixed threshold C, is considered. Boccotti obtained the general solution for $R(C)$, based on the

Rayleigh's linear crest height distribution by applying the 'Equivalent Triangular Storm - ETS' model ([4-14]). The nonlinear return period $R(C)$ for extreme crest height up to the second-order, was then given with a different approach by Arena & Pavone [15] as a function of both the *short-term* and the *long-term* statistics. By applying different *short-term* models exact to the second-order, they proposed also a comparison of the results obtained with their solution for $R(C)$.

In the present paper, the general solution for the nonlinear return period $R(C)$ is applied considering the Dawson crest height model for the *short-term* statistics. The results are then compared with those achieved through second-order crest height distributions related to the condition of both narrow-band and finite bandwidth of the sea spectrum.

2. Nonlinear Models for Crest Height Distributions in a Sea State

By taking into account the effects of non-linearity, which occur regularly at sea, the probability density function of the free surface tends to deviate from the linear Gaussian distribution (to which correspond the Rayleigh law probability distribution for both the crest and the trough amplitudes). In particular, second-order nonlinearities make high crests to be more probable than deep troughs. Therefore, for accurate predictions in the crest height distribution, non-linear wave effects have to be considered. In what follows, some non-linear theoretical models describing the statistics of non-linear wave crests are analysed. Predictions from these non-linear solutions, compared with the linear results, will demonstrate that nonlinearities affect significantly the statistics of large wave crests.

2.1 Narrow-Band Second-Order Crest Height Distribution: the Arena & Fedele Model

Narrow-band models for the second-order distribution of crest heights and trough amplitudes in a sea state, were firstly obtained by Tayfun [6] and Tung and Huang [16]. In particular, Tayfun [6] derived the probability density function and the probability of exceedance of the crest (absolute maximum) for the free surface displacement in an undisturbed wave field, assuming the waves exact to second-order in the Stokes expansion. The probability of exceedance of the second-order trough (absolute minimum) was then achieved by Tung and Huang [16].

A more general approach was then proposed by Arena & Fedele [7], who investigated the statistical properties of a family of narrow-band second-order stochastic processes obtaining both the crest, $P(C;Hs{=}h)$, and the trough, $P(\tau;Hs{=}h)$, distributions of the family. This stochastic family includes many processes of the mechanics of the sea waves, among which the water surface.

In detail, for the second-order free surface displacement in an undisturbed wave field under the assumption of narrow-band spectrum, the probability, proposed by Arena & Fedele [7], that a crest height is greater than C in a sea state with significant wave height, Hs, equal to h, is expressed by

$$P(C;H_S=h)=\exp\left[-\left(1-\sqrt{1+16\vartheta\sqrt{1+4\vartheta^2}\,C/h}\right)^2/8\vartheta^2\right] \tag{1}$$

where ϑ is a parameter of non-linearity, which for narrow-band ocean waves is defined as

$$\vartheta(s, kd) = s[2 + \cosh(2kd)]\cosh(kd)/[16\sinh^3(kd)] \tag{2}$$

with $s=kH_s$ the wave steepness, k the wave number and d the bottom depth. For wind-generated waves the wave steepness ranges typically from 0.22 to 0.30. Note that in deep water ($kd>\pi$) $\vartheta \to s/8$.

2.2 Second-Order Crest Distribution for Three-Dimensional Waves: the Forristall's Model

Through the last decade, a great effort has been addressed to understand the effects of non-linearity on the statistical distributions of three-dimensional sea waves, with distribution of energy among different frequencies and directions.

The most important models, which have been developed for the non-linear crest distribution of three-dimensional waves, are due to Forristall [8] and Prevosto et al. [9]. A different analytical approach, which is valid whichever the spectrum is, was then given by Fedele & Arena [12].

In the present paper, the Forristall model, which gives the probability of exceedance of the crest height (defining the probability that the crest height is larger than C, in a sea state with significant wave height $Hs=h$) through a Weibull law depending on two parameters, is considered; that is

$$P(C; H_S = h) = \exp\left[-\left(\frac{C}{\alpha h}\right)^{\beta}\right]. \tag{3}$$

For three-dimensional waves, the two parameters, α and β, of the law probability (3) are evaluated by the following relations

$$\alpha = 1/\sqrt{8} + 0.2568\, S_1 + 0.0800\, U_r\,; \qquad \beta = 2 - 1.7912\, S_1 - 0.2824\, U_r^2 \tag{4}$$

where S_1 is a parameter which defines the wave steepness, and U_r is the Ursell number on the water depth, d :

$$S_1 = \frac{H_S}{gT_{m01}^2/(2\pi)} \qquad U_r = \frac{H_S}{k_1^2 d^3} \tag{5}$$

with g the acceleration of gravity.

In the previous relations, k_1 is the wave number in deep water

$$k_1 = \frac{2\pi}{gT_{m01}^2/(2\pi)} \qquad \text{with} \qquad T_{m01} = 2\pi\, m_0 / m_1 \tag{6}$$

m_j being the j-th moment of the frequency spectrum.

2.3 Narrow-Band Fifth-Order Crest Height Distribution: the Dawson's Model

The expression of the cumulative probability P that a crest amplitude will be equal or exceed the value C/H_s, for narrow-band sea waves up to the fifth order in a Stokes expansion, has been achieved by Dawson [13]:

$$P(C;H_S=h) = \exp\left[-8\left(\frac{C}{h}\right)^2 + 8\,s'\left(\frac{C}{h}\right)^3 - 4\,s'^2\left(\frac{C}{h}\right)^4 + \frac{14}{3}s'^3\left(\frac{C}{h}\right)^5 - \frac{117}{24}s'^4\left(\frac{C}{h}\right)^6\right] \tag{7}$$

where $H_s=h$ denotes the significant wave height of the waves and s' is a characteristic steepness parameter defined as

$$s' = \beta_0 H_S = \overline{\varpi}^2 H_S / g \tag{8}$$

with $\overline{\varpi}$ the average wave frequency, characterized by the relation that

$$\overline{\varpi} = gk\left(1 + k^2a^2 + \frac{5}{4}k^4a^4\right). \tag{9}$$

3. Return Period of a Sea Storm in which the Maximum Non-Linear Crest Height Exceeds a Fixed Threshold

3.1 The "Equivalent Triangular Storm" Model

A sea storm may be defined as "a sequence of a sea states in which the significant wave height of a sea state, $Hs(t)$, exceeds a fixed threshold h_{crit} and does not fall below this threshold for a continuous time interval greater than a fixed value ΔT_{crit}" (Boccotti [4]). Following Boccotti [4] (see also Arena & Barbaro [17], and Arena [18-19]), we assume the threshold h_{crit} equal to 1.5 times the mean significant wave height and the time span ΔT_{crit} equal to 12 hours.

The Equivalent Triangular Storm (ETS) model, introduced by Boccotti [4-14], defines an Equivalent Triangular Storm for each actual sea storm, which is characterized by the two following parameters:

(i) the height a of the triangle, which is equal to the maximum significant wave height of the actual sea storm and so it is directly achieved;

(ii) the base b of the triangle, which is the duration of the equivalent triangular storm and is such that the maximum wave height expected is equal both in the actual and in the triangular sea storms. It is evaluated through an iterative procedure.

For many storms in several locations, the probability that the maximum wave height in the storm is greater than any fixed threshold was found to be coincident for the actual and the relative triangular sea storm (Boccotti [4], Arena & Barbaro [17], Arena [18-19]). Therefore, even if the real sea storms have time histories generally different from each other, we may extract an ETS that is strongly equivalent to the actual storm.

3.2 The "Equivalent Sea"

By applying the equivalent triangular storm model we obtain the "equivalent sea", which is given by the succession of the equivalent triangular storms (that are then in the same number of actual storms). The peculiarity of the equivalent sea with respect to the actual one is that they have the same probability of exceedance of the significant wave height (Boccotti [3]) for any threshold of significant height.

3.3 The Return Period R(C)

Boccotti [4-14] obtained the analytical solution for the return period $R(H)$ of a sea storm whose maximum wave height exceeds the fixed threshold H. His analytical solution was derived for the 'equivalent sea'.

Here, the Boccotti's logic is applied for the computation of the return period $R(C)$ of a sea storm in which the maximum nonlinear crest height exceeds a fixed threshold C. Considering a very large time interval T, the return period $R(C)$ is defined as

$$R(C) = \frac{\mathrm{T}}{N(C,\mathrm{T};\max)} \tag{10}$$

where $N(C,\mathrm{T};\max)$ is the number of wave crests, during T, which are both higher than C and the highest in their own storm. This number is given by

$$\begin{aligned} N(C,\mathrm{T};\max) = -\mathrm{T} \int_C^\infty \int_0^\infty \frac{p(C';H_S=h)}{1\text{-}P(C';H_S=h)} \frac{1}{\overline{T}(h)} \int_h^\infty \frac{1}{\overline{b}(a)} \frac{\mathrm{d}p(H_S=a)}{\mathrm{d}a} \\ \cdot \int_0^\infty b p_b(b \mid a) \exp\left\{ \frac{b}{a} \int_0^a \frac{\ln[1-P(C';H_S=h')]}{\overline{T}(h')} \mathrm{d}h' \right\} \mathrm{d}b \, \mathrm{d}a \, \mathrm{d}h \, \mathrm{d}C'. \end{aligned} \tag{11}$$

If we assume the unknown conditional probability density function, of the triangle base given the triangle height $p_b(b \mid a)$, expressed by the delta function:

$$p_b(b \mid a) = \delta[b - \overline{b}(a)] \tag{12}$$

we have that

$$\int_0^\infty p_b(b \mid a)\, b \exp[f(a,C')b] \, \mathrm{d}b = \overline{b}(a) \exp[f(a,C')\overline{b}(a)] \tag{13}$$

by which the following analytical expression of $R(C)$ is achieved:

$$\begin{aligned} R(C) = \Bigg\{ - \int_C^\infty \int_0^\infty \frac{p(C';H_S=h)}{1\text{-}P(C';H_S=h)} \frac{1}{\overline{T}(h)} \int_h^\infty \frac{\mathrm{d}p(H_S=a)}{\mathrm{d}a} \\ \cdot \exp\left[\frac{\overline{b}(a)}{a} \int_0^a \frac{\ln[1-P(C';H_S=h')]}{\overline{T}(h')} \mathrm{d}h' \right] \mathrm{d}a \, \mathrm{d}h \, \mathrm{d}C' \Bigg\}^{-1}. \end{aligned} \tag{14}$$

The assumption (13) of the delta function to represent the $p_b(b \mid a)$ is a little conservative [with respect to different forms of $p_b(b \mid a)$], as it was shown by Boccotti [4] (see also Arena & Pavone [15]). Anyway, for any fixed value R' of the return period, the differences for the value $C(R')$ are smaller than 2%.

In what follows, the non-linear $R(C)$ will be computed through Eq. (14) by considering one of the nonlinear crest height distributions presented in the previous Sections.

4. Results and Discussion

In this section, the return periods $R(C)$ is calculated by processing buoy data at two different locations. The former is Ponza, in the Central Mediterranean Sea, which belongs to the Italian Wave Measurement Network (RON of the APAT – Italian Agency for the Environment and Technical Survey). The latter is in the Pacific Ocean with reference to the buoy 46002 of the USA National Data Buoy Center (NOAA-NDBC). As regard to the *long-term* statistics, expressions and data are assumed from Arena [18]. In particular, for the regression base-height of the equivalent triangular storms, $\bar{b}(a)$, is considered an exponential law and the probability density function $p(Hs=a)$ is represented by a Weibull law (Boccotti, [4]; Arena & Barbaro, [17]).

4.1 Nonlinear Predictions obtained through Second-Order Models

Nonlinear effects on the crest height distribution in a sea state are analysed. As second-order effect, the probability density function of the free surface tends to deviate from the linear Gaussian distribution giving that the higher crest heights are underestimated by linear Rayleigh law.

In particular, considering the second-order narrow-band models of Arena & Fedele [(1)] and of Dawson [(7)], retaining the terms up to the second-order, a quite good agreement is achieved (see Figure 1 – upper panel). For fixing value of the probability of exceedance, the height of the extreme second-order crest amplitude gets through the Dawson model is slightly greater than that one achieved by the Arena & Fedele distribution.

Moreover, it is found that (Figure 1 – upper panel) narrow-band models are conservative with respect to the Forristall distribution, which takes into account the effect of three-dimensionality of random sea waves. In words, the finite bandwidth of the spectrum (by considering both the frequency spectrum as well as the directional spreading function) predicts a little reduction of the highest crest with respect to second-order narrow-band models.

Therefore, the linear and the nonlinear return period $R(C)$, evaluated through the different second-order models for the wave crest distribution, is shown in Figures 2 and 3 (upper panels) for the two examined locations. In both cases, predictions obtained from the second-order narrow-band model are conservative with respect to those taking into account the effects of finite bandwidth and of directional spreading.

4.2 Effects of Higher Order Terms

Nonlinear effects, up to the fifth order, are then analysed by considering the complete solution of Dawson model. The results are shown in Figure 1 (lower panel) for the $P(C;\ Hs=h)$ and in Figures 2 and 3 (lower panels) for the calculation of the return period $R(C)$.

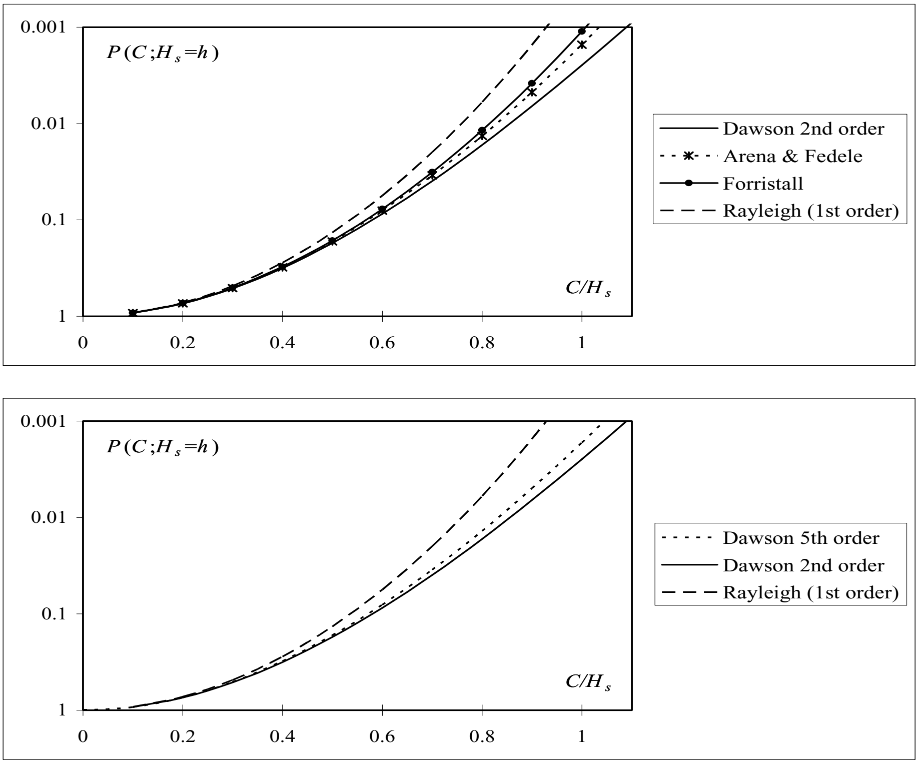
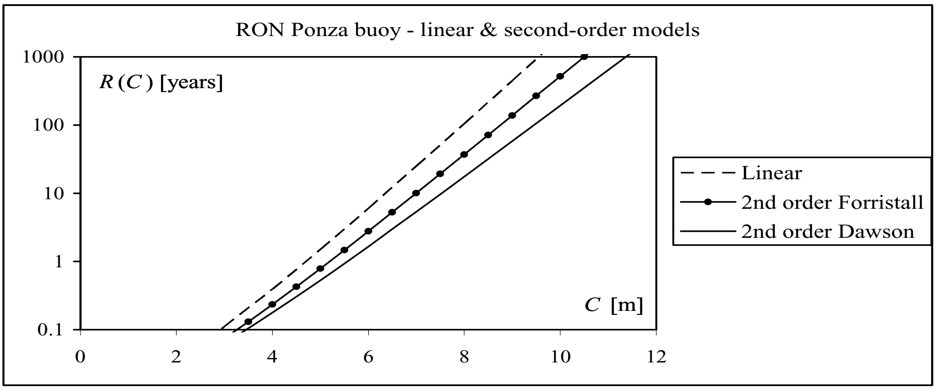


Figure 1. The probability of exceedance of crest heights. Comparison among the linear Rayleigh law and the second-order distributions of Dawson (narrow-band model), Arena & Fedele (narrow-band model) and Forristall (distribution for three-dimensional waves): upper panel; comparison among the linear Rayleigh law and second-order and fifth-order narrow-band Dawson distribution: lower panel.



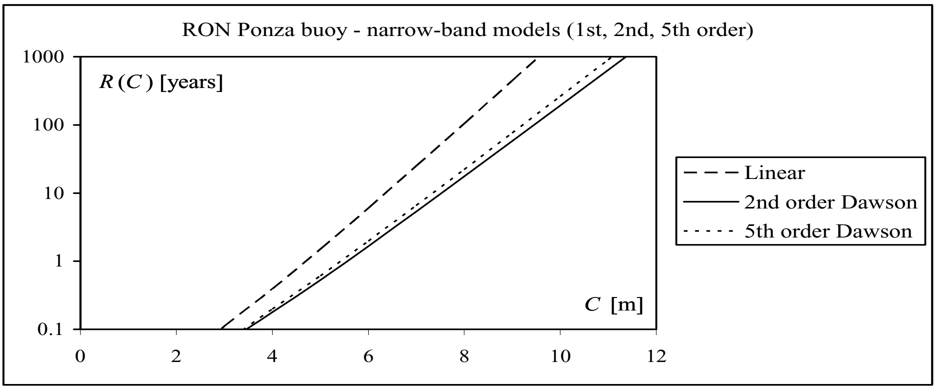


Figure 2. The return period of a sea storm in which the maximum non-linear crest height exceeds a fixed threshold, calculated from data of NOAA-NDBC 46002 buoy (of the USA buoys network) moored in the Pacific Ocean. Comparison among prediction given by: linear model, second and fifth order narrow-band Dawson model (lower panel), Forristall second-order model for three-dimensional waves and second-order narrow-band Dawson model (upper panel).

The conclusion is that the extreme crest heights during storms are maxima by considering the second-order effects and narrow-band condition of the wave frequency spectrum: the fifth-order solution shows that extreme crests are slightly smaller than second-order values.

Nevertheless, by taking into account the effects both of finite bandwidth of the spectrum and of the directional spreading, which means to analyse the more realistic feature of three-dimensional sea waves, second-order predictions are the least restrictive.

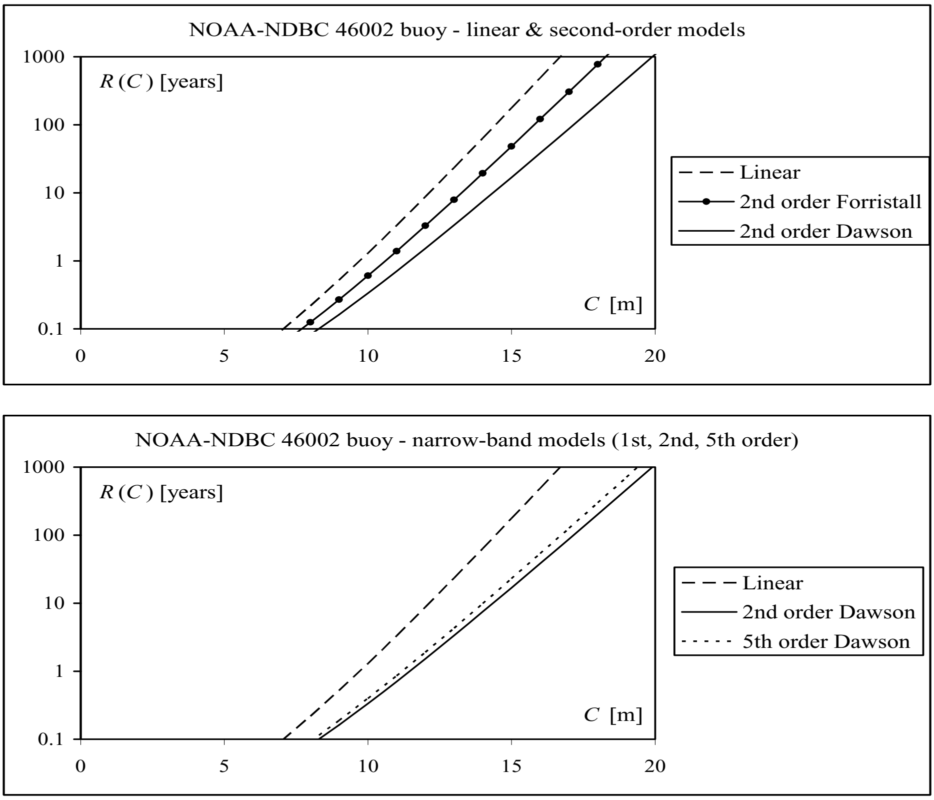


Figure 3. The return period of a sea storm in which the maximum non-linear crest height exceeds a fixed threshold, calculated from data of Ponza RON buoy (of the Italian buoys network of the APAT) moored in the Central Mediterranean Sea. Comparison among prediction given by: linear model, second and fifth order narrow-band Dawson model (lower panel), Forristall second-order model for three-dimensional waves and second-order narrow-band Dawson model (upper panel).

5. Conclusions

In this paper the nonlinear solution for the return period $R(C)$ of a crest height which is maximum in its own storm and exceeds a threshold C is analysed, by considering the short-term crest statistics up to the fifth order in a Stokes expansion.

Firstly, the second-order effects are analysed. It has been obtained that the Dawson model predicts extreme crest heights slightly greater than those obtained through the Arena & Fedele distribution and that both narrow-band models are slightly conservative with respect to the results of the Forristall probability of exceedance for three-dimensional waves.

Finally, including the non-linearity up to the fifth-order for the narrow-band condition of the frequency spectrum, the extreme crests are slightly reduced with respect to the second-order models.

Therefore, for engineering applications, the Dawson second-order model may be applied in place of the higher-order solution obtaining results which are just a little conservative.

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POWERS OF MIXED PRODUCTS IDEALS AND NORMALITY OF THE ASSOCIATED MONOMIAL ALGEBRA

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Abstract: Let $L = I_2 + J_2$ be the ideal of mixed products, being $I_2 = (\{x_i x_j, 1 \leq i < j \leq n\})$ and $J_2 = (\{y_l y_k, 1 \leq l < k \leq m\})$ ideals of $K[x_1, x_2, ..., x_n; y_1, y_2, ..., y_m]$, the polynomial ring in two disjoint sets of variables. We study the normality of the toric ideal P of $K[L^i]$, $i = 1, 2$, determining a Gröbner basis for P.
Keywords: Monomial algebras, toric ideals, Gröbner basis.
Classification AMS: 13B22, 13F20, 13P10, 13A30.

Introduction

The study of the normality of a semigroup ring A is directly connected to the normality of the associated toric variety. Combinatoric methods support these results. More exactly, if P is the toric ideal of A and $A = B/P$, P is always generated by binomials of B. If a total term order is introduced among the monomials of B, and $in_<(P)$ is a square-free ideal, then the toric variety defined by P is normal ([3] Th. 13.15). This result has been used in [4] (Th. 7.5.8) to prove the normality of the semigroup ring $K[L]$, being L a mixed products ideal of the polynomial ring $K[x_1, x_2, ..., x_n; y_1, y_2, ..., y_m]$ in two disjoint sets of variables and K a field of characteristic 0. This class of ideals have been introduced in [2], where the normality has been studied. If L is normal, then $K[L]$ is normal ([4], Prop. 3.3.18, Prop. 7.4.1). We want to study the problem of the normality of $K[L^i]$, being L^i the i-power of L, when L is not normal. If $i = 1$, L is always integrally closed since L is monomial ideal square-free ([4], Cor 7.3.15). For $i > 1$, L^i can not be integrally closed. In this work, we consider the ideal $L = I_2 + J_2$ of the polynomial ring $K[\underline{x}; \underline{y}] = K[x_1, x_2, ..., x_n; y_1, y_2, ..., y_m]$, where $I_2 = (\{x_i x_j, 1 \leq i < j \leq n\})$ and $J_2 = (\{y_l y_k, 1 \leq l < k \leq m\})$. We observe that L is the edge ideal associated to the tensor product of two complete graphs with n and m vertices respectively, $G_1 \bigotimes G_2$. The ideal L is not normal and the first power not integrally closed is L^3 ([4], Prop. 7.5.7). The study of the normality of $K[L^i]$, $i \geq 2$ is an open problem. In N°1, we study the normality of $K[L]$, being $L = I_2 + J_2$, ideal of mixed products of $K\left[\underline{x};\underline{y}\right]$. The result is proved in [1] but we complete the previous result calculating a Gröbner basis for the toric ideal P of $K[L]$. In N°2, we study the normality of $K\left[L^2\right]$. A particular term order, called *sort order* ([3], Chapter 14), is introduced to calculate a Gröbner basis for the toric ideal P of $K\left[L^2\right]$.

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1 Normality of $K[L]$

Let $R = K[x_1, x_2, ..., x_n]$ be the polynomial ring over a field K and I_2 be the ideal of R generated by all the square-free monomials of degree two,

$$I_2 = (\{x_i x_j,\ 1 \leq i < j \leq n\})$$

Set $f_{ij} = x_i x_j$, then $K[I_2] = K[\{f_{ij}, 1 \leq i < j \leq n\}]$ is the subring of R generated by f_{ij}. There is an homomorphism

$$\varphi : K[\{T_{ij}, 1 \leq i < j \leq n\}] \longrightarrow K[I_2]$$

induced by $T_{ij} \longrightarrow f_{ij}$.
The ideal $P_2 = Ker(\varphi)$ is the **presentation ideal** or **toric ideal** of $K[I_2]$.

Theorem 1.1 *Let $R = K[x_1, x_2, ..., x_n]$ be the polynomial ring over a field K and P_2 the toric ideal of $K[I_2]$. Set*

$$B = \{T_{ij}T_{kl} - T_{il}T_{jk}, T_{ik}T_{jl} - T_{il}T_{jk}, 1 \leq i < j < k < l \leq n\}$$

and

$$T_{12} < T_{13} < ... < T_{1n} < T_{23} < ... < T_{2n} < ... < T_{(n-1)n}$$

is a term order for the variables, then B is a ***reduced Gröbner basis*** *for the toric ideal P_2.*

Proof: ([4], Prop. 9.2.1).

Corollary 1.1 *Let $in_<(P_2)$ be the initial ideal of P_2, then $in_<(P_2)$ is square-free of degree two and it is generated by the following leading monomials:*

$$in_<(P_2) = (\{T_{ij}T_{kl}, T_{ik}T_{jl}, 1 \leq i < j < k < l \leq n\})$$

Proof: By Theorem 1.1, choosing lexicographic order.

Corollary 1.2 *$K[I_2]$ is normal.*

Proof: ([3], Th. 13.15).

Now, consider two sets of variables $\underline{x} = (x_1, x_2, ..., x_n)$, $\underline{y} = (y_1, y_2, ..., y_m)$ and the polynomial ring $\mathbb{A} = K[x_1, x_2, ..., x_n; y_1, y_2, ..., y_m] = K\left[\underline{x}; \underline{y}\right]$.

Definition 1.1 *Let k, r, s, t be integers, non negative, $k, r, s, t \geq 0$, such that $k + r = s + t$.*

$$L = I_k J_r + I_s J_t$$

is the mixed products ideal of $\mathbb{A}$, where I_k (resp. J_r) is the ideal generated by all the square-free monomials of degree k (resp. r) in the variables x_i (resp. y_j).

The only mixed products ideals that are normal are ([2]):

(i) $L = I_k J_r + I_{k+1} J_{r-1}$, $k \geq 0$, $r \geq 1$

(ii) $L = I_k J_t$, $k \geq 1$, $t \geq 1$

(iii) $L = J_r + I_m J_t$, and $L = I_k + J_m I_t$

Theorem 1.2 *Let $\mathbb{A} = K[x_1, x_2, ..., x_n; y_1, y_2, ..., y_m]$ be the polynomial ring in two disjoint sets of variables x_i, $i = 1, ..., n$, y_l, $l = 1, ..., m$, over a field K. Let*

$$L = I_2 + J_2$$

Set $f_{ij} = x_i x_j$ and $g_{lk} = y_l y_k$, then

$$K[L] = K[\{f_{ij}, g_{lk}, 1 \leq i < j \leq n, 1 \leq l < k \leq m\}]$$

is the monomial subring of $\mathbb{A}$ generated by f_{ij} and g_{lk}. We consider the homomorphism

$$\phi : K[\{T_{ij}, U_{lk}, 1 \leq i < j \leq n, 1 \leq l < k \leq m\}] \longrightarrow K[L]$$

$$T_{ij} \longrightarrow f_{ij}, U_{lk} \longrightarrow g_{lk}$$

Let $P = Ker(\phi)$ be the toric ideal of $K[L]$. Set

$$B^{'} = \{T_{ij}T_{rs} - T_{is}T_{jr}, T_{ir}T_{js} - T_{is}T_{jr}, 1 \leq i < j < r < s \leq n\}$$

and

$$B^{''}\left\{U_{lk}U_{vz}-U_{lz}U_{kv},U_{lv}U_{kz}-U_{lz}U_{kv},1\leq l<k<v<z\leq m\right\}$$

If the terms of $B=B^{'}\cup B^{''}$ *are ordered by*

$$T_{12}<T_{13}<...<T_{1n}<T_{23}<...<T_{2n}<...<T_{(n-1)n}<U_{12}<U_{13}...<...<U_{(m-1)m}$$

then B *is a* ***minimal generating set*** *for* P *and it is a* ***reduced Gröbner*** *for* P *with the lex order induced on the monomials of* $K\left[T_{ij},U_{lk}\right]$.

Proof: Let f and g be two elements of $B^{'}\cup B^{''}$. To prove that B is a reduced Gröbner basis, we use the S-pairs method.

(i) if $f,g\in B^{'}$ then is true by theorem 1.1.

(ii) if $f,g\in B^{''}$ then is true by theorem 1.1.

(iii) if $f\in B^{'}$ and $g\in B^{''}$, let $lm\left(f\right)$ be *the leading monomial* of f and $lm\left(g\right)$ be *the leading monomial* of g. Since $lm\left(f\right)$ and $lm\left(g\right)$ are co-primes then:
$S\left(f,g\right)\longrightarrow_{B^{'}\cup B^{''}}0$. ([4], Lemma 2.4.14)

Corollary 1.3 *Since* $G=B^{'}\cup B^{''}$ *is a reduced Gröbner basis for* P*, then* $in_{<}\left(P\right)$ *is square-free.*

Proof:
$in_{<}\left(P\right)=\left(T_{ij}T_{rs},T_{ir}T_{js},U_{lk}U_{vz},U_{lv}U_{kz},1\leq i<j<r<s\leq n,1\leq l<k<v<z\leq m\right)$.

Corollary 1.4 $K\left[L\right]$ *is normal.*

Proof: Since the toric ideal P of $K\left[L\right]$ has a square-free initial ideal, $in_{<}\left(P\right)$, then $K\left[L\right]$ is normal. ([3], Th 13.15)

Remark 1.1 *It is well known that if* L *is normal, then* $K\left[L\right]$ *is normal. In fact if* L *is normal then the Rees Algebra* $\Re\left[L\right]$ *is normal* ([4], Prop. 3.3.18).$K\left[L\right]$ *is normal, since* $K\left[L\right]$ *is the fiber of the Rees Algebra,* $\Re\left[L\right]$*, in the closed point* ([4], Prop. 7.4.1).

Questions: Let L be a mixed products ideal. The general requests are the following:

- $K[L]$ is normal?
- $K\left[L^i\right]$ is normal, being L^i the i-power of L?

Since L is always integrally closed ([4], Th. 7.4.5), being a square-free ideal, a sufficient condition to obtain $K[L]$ normal is that L is integrally closed ([1]). In general, the powers L^i of L aren't integrally closed, when L is not normal. Then, the problems are:

1. If L^i is integrally closed, then $K[L]$ is normal?
2. Even if L^i is not integrally closed, then $K\left[L^i\right]$ is normal?

2 Normality of $K\left[L^2\right]$

We fix positive integers r and $s_1, s_2, ..., s_d$. We consider the set

$$\mathcal{A} = \left\{(i_1, i_2, ..., i_d) \in \mathbb{Z}^d, i_1 + i_2 + ... + i_d = r, 0 \leq i_1 \leq s_1, \ldots, 0 \leq i_d \leq s_d\right\}$$

There is a natural bijection between the elements of $\mathcal{A}$ and weakly increasing strings of length r over the alphabet $\{1, 2, .., d\}$. Under this bijection, the vectors $(i_1, i_2, ..., i_d) \in \mathcal{A}$ are mapped to the weakly increasing string

$$\underline{u} = u_1 u_2 \cdots u_r = \underbrace{11\cdots 1}_{i_1 volte}\underbrace{22\cdots 2}_{i_2 volte}\cdots\underbrace{dd\cdots d}_{i_d volte}$$

Set $T_{u_1 u_2 \cdots u_r}$ the corresponding variable in the polynomial ring $K[\underline{T}]$. Let $sort(\cdot)$ the operator which orders any string $\{1, 2, .., d\}$ into weakly increasing order. Using this convention the toric ideal is:

$$I_{\mathcal{A}} = \left(\{T_{\underline{u}} T_{\underline{v}} \cdots T_{\underline{w}} - T_{\underline{u}'} T_{\underline{v}'} \cdots T_{\underline{w}'}, sort(\underline{uv} \ldots \underline{w}) = sort\left(\underline{u}'\underline{v}' \cdots \underline{w}'\right)\}\right).$$

A monomial $T_{u_1 u_2 \cdots u_r} T_{v_1 v_2 \cdots v_r} \cdots T_{w_1 w_2 \cdots w_r} \in K[\underline{T}]$ is said to be sorted if

$$u_1 \leq v_1 \leq \cdots \leq w_1 \leq u_2 \leq v_2 \cdots \leq w_2 \leq u_3 \leq \cdots \leq u_d \leq v_d \leq \cdots w_d$$

Theorem 2.1 *There exits a term order $<$ on $K[\underline{T}]$ such that the sorted monomials are precisely the $<$-standard monomials modulo $I_{\mathcal{A}}$. The initial ideal $in_<(I_{\mathcal{A}})$ is generated by square-free quadratic monomials. The corresponding reduced Gröbner basis for $I_{\mathcal{A}}$ is*

$$\{T_{u_1 u_2 \cdots u_r} T_{v_1 v_2 \cdots v_r} - T_{w_1 w_3 \cdots w_{2r-1}} T_{w_2 w_4 \cdots w_{2r}}, w_1 w_2 w_3 w_4 \cdots w_{2r} = sort(u_1 v_1 u_2 v_2 \cdots$$

Proof: ([3], Th. 14.2)

Let $R = K[x_1, x_2, ..., x_n]$ be the polynomial ring over the field K.

Definition 2.1 *An ideal $I_{k,t}$ of R is a Veronese type ideal if it is generated by all the monomials of degree k and any variable has maximum degree equal to t.*

$$I_{k,t} = \left(\left\{x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}, \sum_{s=1}^{n} i_s = k, 0 \le i_s \le t\right\}\right)$$

Let

$$I_2 = (\{x_i x_j, 1 \le i < j \le n\})$$

be the ideal of R generated by all the square-free monomials of degree two and

$$I_2^2 = \left(\{x_i^{i_1}x_j^{i_2}x_k^{i_3}x_l^{i_4}, 1 \le i < j < k < l \le n, i_1 + i_2 + i_3 + i_4 = 4, 0 \le i_s \le 2, s = 1, 2, 3, 4\}\right)$$

Proposition 2.1 *The ideals I_2 and I_2^2 are Veronese type ideals.*

Proof: By Definition 2.1.

Example 2.1 *We consider $R = K[x_1, x_2, x_3]$.*

$$I_2 = (x_1x_2, x_1x_3, x_2x_3)$$

$$I_2^2 = \left(x_1^2x_2^2, x_1^2x_2x_3, x_1x_2^2x_3, x_1^2x_3^2, x_1x_2x_3^2, x_2^2x_3^2\right)$$

$$\mathcal{A}_{I_2^2} = \{(2,2,0), (2,1,1), (1,2,1), (2,0,2), (1,1,2), (0,2,2)\}$$

with $s_1 = s_2 = s_3 = 2$.
Let $\underline{u} = 1122$, $\underline{v} = 1123$, $\underline{z} = 1223$, $\underline{w} = 1133$, $\underline{c} = 1233$, $\underline{h} = 2233$ the weakly increasing string, then the toric ideal is

$$I_{\mathcal{A}_{I_2^2}} = (T_{1233}T_{1233} - \underline{T_{1133}T_{2233}}, T_{1223}T_{1233} - \underline{T_{1123}T_{2233}}, \underline{T_{1223}T_{1133}} - T_{1123}T_{1233},$$

$$T_{1223}T_{1223} - \underline{T_{1122}T_{2233}}, T_{1123}T_{1223} - \underline{T_{1122}T_{1233}}, T_{1123}T_{1123} - \underline{T_{1122}T_{1133}})$$

where the marked monomials are the initial terms. Since

$$in_<(I_{\mathcal{A}_{I_2^2}}) = (T_{1133}T_{2233}, T_{1123}T_{2233}, T_{1223}T_{1133}, T_{1122}T_{2233}, T_{1122}T_{1233}, T_{1122}T_{1133})$$

is generated by square-free monomials, then $K[I_2]$ is normal.

Proposition 2.2 *Let $\mathbb{A} = K[x_1, x_2, ..., x_n; y_1, y_2, ..., y_m]$ be the polynomial ring in two disjoint sets of variables x_i, $i = 1, ..., n$, y_l, $l = 1, ..., m$, over a field K, let $L = I_2 + J_2$. Then the ideal $L^2 = I_2^2 + J_2^2 + I_2 J_2$ isn't a Veronese type ideal.*

Proof: The monomials $x_i^2 y_l y_t$, $x_i x_j y_l^2 \notin L^2$, for $i, j = 1, 2, \ldots, n$, $l, t = 1, 2, \ldots, m$

Theorem 2.2 *Let $\mathbb{A} = K[x_1, x_2, ..., x_n; y_1, y_2, ..., y_m]$ be a polynomial ring in two disjoint sets of variables and let $L = I_2 + J_2$ be the mixed products ideal with $I_2 = (\{x_i x_j, 1 \leq i < j \leq n\})$ and $J_2 = (\{y_l y_t, 1 \leq l < t \leq m\})$. Then $K\left[L^2\right]$ has a square-free Gröbner basis.*

Proof: The binomials which are in the toric ideal $I_{\mathcal{A}_{L^2}}$ are of the following type:

1) $T_{\underline{u}_i} T_{\underline{u}_j} - T_{\underline{u}_a} T_{\underline{u}_b} \in I_2^2$ such that sort $\left(\underline{u}_i \underline{u}_j\right)$ =sort $(\underline{u}_a \underline{u}_b)$

2) $T_{\underline{v}_k} T_{\underline{v}_h} - T_{\underline{v}_c} T_{\underline{v}_d} \in J_2^2$ such that sort $(\underline{v}_k \underline{v}_h)$ =sort $(\underline{v}_c \underline{v}_d)$

3) $T_{\underline{u}_i} T_{\underline{r}_l;\underline{s}_m} - T_{\underline{u}_k} T_{\underline{r}_n;\underline{s}_m}$ such that sort $(\underline{u}_i \underline{r}_l)$ =sort $(\underline{u}_k \underline{r}_n)$

4) $T_{\underline{v}_k} T_{\underline{r}_p;\underline{s}_q} - T_{\underline{v}_w} T_{\underline{r}_p;\underline{s}_e}$ such that sort $\left(\underline{v}_k \underline{s}_q\right)$ =sort $(\underline{v}_w \underline{s}_e)$

5) $T_{\underline{r}_l;\underline{s}_m} T_{\underline{r}_n;\underline{s}_p} - T_{\underline{r}_l;\underline{s}_p} T_{\underline{r}_n;\underline{s}_m} \in I_2 J_2$

6) $T_{\underline{r}_p;\underline{s}_m} T_{\underline{r}_q;\underline{s}_n} - T_{\underline{u}_i} T_{\underline{v}_k}$ such that sort $\left(\underline{r}_p \underline{r}_q\right) = \underline{u}_i$ and sort $(\underline{s}_m \underline{s}_n) = \underline{v}_k$

To prove that this set is a Gröbner basis for $I_{\mathcal{A}_{L^2}}$, we consider the S-pair method. If $f, g \in I_{\mathcal{A}_{L^2}}$ are of the type $T_{\underline{u}_i} T_{\underline{u}_j} - T_{\underline{u}_a} T_{\underline{u}_b}$ then, the set that they generate is a Gröbner basis for the ideal that they generate using the sort order. Likewise for the binomials of type 2), 5), 6). Let $f = T_{\underline{u}_i} T_{\underline{r}_l;\underline{s}_m} - T_{\underline{u}_k} T_{\underline{r}_n;\underline{s}_m}$ a binomial of type 3). We fix the following term order:
let $g = T_{\underline{u}_i} T_{\underline{u}_j} - T_{\underline{u}_a} T_{\underline{u}_b}$, if $T_{\underline{u}_i} T_{\underline{u}_j}$ is the leading term of g then we choose

- $\underline{u}_k = \underline{u}_a$
- sort $(\underline{r}_l \underline{r}_n) = \underline{u}_b$
- sort$\left(\underline{u}_i \underline{u}_j\right)$ =sort$(\underline{u}_a \underline{u}_b)$
- $T_{\underline{u}_i} T_{\underline{r}_l;\underline{s}_m}$ as leading term of g.

Then $S(f,g) = T_{\underline{u}_a}\left(T_{\underline{u}_b}T_{\underline{r}_l;\underline{s}_m} - T_{\underline{u}_j}T_{\underline{r}_n;\underline{s}_m}\right)$ is reduced to zero by the binomials of type 3).
Likewise, let $h = T_{\underline{v}_k}T_{\underline{r}_p;\underline{s}_q} - T_{\underline{v}_w}T_{\underline{r}_p;\underline{s}_e}$ a binomial of type 4). Let $k = T_{\underline{v}_k}T_{\underline{v}_h} - T_{\underline{v}_c}T_{\underline{v}_d}$, if $T_{\underline{v}_k}T_{\underline{v}_h}$ is the leading term of k, then we choose:

- $\underline{v}_c = \underline{v}_w$
- sort $\left(\underline{s}_q\underline{s}_e\right) = \underline{v}_d$
- sort$(\underline{v}_k\underline{v}_h)$ =sort$(\underline{v}_c\underline{v}_d)$
- $T_{\underline{v}_k}T_{\underline{r}_p;\underline{s}_q}$ as leading term of h

Then $S(h,k) = T_{\underline{v}_c}\left(T_{\underline{v}_d}T_{\underline{r}_p;\underline{s}_q} - T_{\underline{v}_h}T_{\underline{r}_p;\underline{s}_e}\right)$ is reduced to zero by binomials of type 4).

Corollary 2.1 *$K\left[L^2\right]$ is normal*

Proof: By Theorem 2.2 $in_<(I_{\mathcal{A}_{L^2}})$ is generated by square-free monomials of degree two. ([3], Th. 13.15)

Example 2.2 *Let $\mathbb{A} = K[x_1, x_2, x_3; y_1, y_2, y_3]$ be the polynomial ring in two disjoint sets of variables x_i and y_l, $i = 1,2,3$ and $l = 1,2,3$. Let*

$$I_2 = (x_1x_2, x_1x_3, x_2x_3)$$

$$J_2 = (y_1y_2, y_1y_3, y_2y_3)$$

$$L = I_2 + J_2$$

$$L^2 = I_2^2 + J_2^2 + I_2J_2$$

then $K[I_2^2]$ is normal, and, likewise, $K[J_2^2]$ is normal. By ([1], Th. 2.4) *$K[I_2J_2]$ is normal. In fact, set $y_1 = x_4$, $y_2 = x_5$, $y_3 = x_6$ we have:*

$$I_2J_2 = (x_1x_2x_4x_5, x_1x_2x_4x_6, x_1x_2x_5x_6, x_1x_3x_4x_5, x_1x_3x_4x_6,$$

$$x_1x_3x_5x_6, x_2x_3x_4x_5, x_2x_3x_4x_6, x_2x_3x_5x_6)$$

$$\mathcal{A}_{I_2J_2} = \{(1,1,0,1,1,0), (1,1,0,1,0,1), (1,1,0,0,1,1),$$

$$(1,0,1,1,1,0), (1,0,1,1,0,1), (1,0,1,0,1,1), (0,1,1,1,1,0), (0,1,1,1,0,1), (0,1,1,0,1,1)\}$$

Let

$$T_{1245}, T_{1246}, T_{1256}, T_{1345}, T_{1346}, T_{1356}, T_{2345}, T_{2346}, T_{2356}$$

be the corresponding variables in $K[\underline{T}]$, *then*

$$I_{\mathcal{A}_{I_2J_2}} = (\underline{T_{1356}T_{2346}} - T_{1346}T_{2356}, \underline{T_{1256}T_{2346}} - T_{1246}T_{2356}, \underline{T_{1356}T_{2345}} - T_{1345}T_{2356},$$

$$\underline{T_{1256}T_{2345}} - T_{1245}T_{2356}, \underline{T_{1346}T_{2345}} - T_{1345}T_{2346}, \underline{T_{1246}T_{2345}} - T_{1245}T_{2346},$$

$$\underline{T_{1256}T_{1346}} - T_{1246}T_{1356}, \underline{T_{1256}T_{1345}} - T_{1245}T_{1356}, \underline{T_{1246}T_{1345}} - T_{1245}T_{1346})$$

Since $in_<(I_{\mathcal{A}_{I_2J_2}})$ *is generated by square-free monomials, then* $K[I_2J_2]$ *is normal.*
Let

$$L^2 = (x_1^2x_2^2, x_1^2x_2x_3, x_1x_2^2x_3, x_1^2x_3^2, x_1x_2x_3^2, x_2^2x_3^2, x_4^2x_5^2, x_4^2x_5x_6, x_4x_5^2x_6, x_4^2x_6^2, x_4x_5x_6^2,$$

$$x_5^2x_6^2, x_1x_2x_4x_5, x_1x_2x_4x_6, x_1x_2x_5x_6, x_1x_3x_4x_5, x_1x_3x_4x_6, x_1x_3x_5x_6, x_2x_3x_4x_5, x_2x_3x_4x_6, x_2x_3x_5x_6)$$

$$\mathcal{A}_{L^2} = \mathcal{A}_{I_2^2} \cup \mathcal{A}_{J_2^2} \cup \mathcal{A}_{I_2J_2} = \{(i_1, i_2, i_3; 0, 0, 0), (0, 0, 0; k_1, k_2, k_3), (l_1, l_2, l_3; m_1, m_2, m_3)\}$$

$i_1 + i_2 + i_3 = 4,\ 0 \le i_1, i_2, i_3 \le 2$
$k_1 + k_2 + k_3 = 4,\ 0 \le k_1, k_2, k_3 \le 2$
$l_1 + l_2 + l_3 = 2,\ 0 \le l_1, l_2, l_3 \le 1$
$m_1 + m_2 + m_3 = 2,\ 0 \le m_1, m_2, m_3 \le 1$

$$\mathcal{A}_{L^2} = \{(2,2,0;0,0,0), (2,1,1;0,0,0), (1,2,1;0,0,0), (2,0,2;0,0,0), (1,1,2;0,0,0),$$

$$(0,2,2;0,0,0), (0,0,0;2,2,0), (0,0,0;2,1,1), (0,0,0;1,2,1), (0,0,0;2,0,2), (0,0,0;1,1,2),$$

$$(0,0,0;0,2,2), (1,1,0;1,1,0), (1,1,0;1,0,1), (1,1,0;0,1,1), (1,0,1;1,1,0), (1,0,1;1,0,1),$$

$$(1,0,1;0,1,1), (0,1,1;1,1,0), (0,1,1;1,0,1), (0,1,1;0,1,1)\}$$

In according to Theorem 2.2, *the binomial of the toric ideal of* $I_{\mathcal{A}_{L^2}}$ *are of the type* $1), 2), 3), 4), 5), 6)$. *For the binomials of type* $1), 2), 5), 6)$, *we determine the initial term using the sort order, i.e:*

$$T_{1123}T_{1233} - \underline{T_{1223}T_{1133}},$$

$$\underline{T_{4466}T_{5566}} - T_{4566}T_{4566},$$

$$T_{13;46}T_{23;56} - \underline{T_{13;56}T_{23;46}},$$

$$T_{12;45}T_{13;45} - \underline{T_{1123}T_{4455}}.$$

where the marked monomials are the leadings terms.
For the binomials of type 3)*, i.e.* $T_{1123}T_{23;56} - T_{1223}T_{13;56}$*, we consider the following S-couple:*

$$S\left(T_{1123}T_{23;56} - T_{1223}T_{13;56}, \underline{T_{1123}T_{2233}} - T_{1223}T_{1233}\right) = T_{1223}\left(T_{23;56}T_{1233} - T_{13;56}T_{2233}\right)$$

choosing the monomial $T_{1123}T_{23;56}$ *as leading term, in according to Theorem* 2.2*. Likewise for the binomials of type* 4)*, i.e.* $T_{4556}T_{13;46} - T_{4456}T_{13;56}$*, we consider the following S-couple:*

$$S\left(T_{4556}T_{13;46} - T_{4456}T_{13;56}, T_{4556}T_{4466} - T_{4456}T_{4566}\right) = T_{4456}\left(T_{4566}T_{13;46} - T_{4466}T_{13;56}\right)$$

where $T_{4556}T_{13;46}$ *is chosen as leading term.*
By these choices, $in_<(I_{\mathcal{A}_{L^2}})$ *is generated by square-free monomials and then* $K\left[L^2\right]$ *is normal.*

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Identification of the shape-type of randomly deformed axial-symmetric particles

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ABSTRACT

We consider the situation where a set of similarly looking particles is analysed and where it may be assumed that the observed particles have been obtained from a common axial-symmetric parametric prototype form class by individual random deformation. An identification rule is presented, which allows to choose between two different given shape-types based on a few easily to perform measurements. It turns out that the misclassification probabilities may be evaluated asymptotically provided that the number of particles in the set is large.

INTRODUCTION

Quite often the outcome of an experimental investigation presents itself in the form of a set of different, disjoint, but to a certain extent similar particles in an Euclidean space $\mathcal{R}^k$. The near similarity of the particles suggests then to assume that the particles have arisen by individual random deformation applied to the contours of forms taken from a parametric class of bodies, which represents the shape-type of the typical particle. When constructing a suitable stochastic model explaining the genesis of the experimental outcome, an important task consists in reconstructing the original form of the particles. In the following we treat problems where a choice has to be made between two different prototype shapes. Based on a few easily to perform measurements the perferred shape-type has to be selected in such a way that in a certain sense the unfavorable effects of an incorrect shape-type attribution are minimized. This leads to an application of the techniques of discriminant analysis in the context of geometrical stochastics.

ON IDENTIFICATION RULES

In the problem considered two particle populations G_1 and G_2 are at disposal for selecting the original forms of the particles of the set U under examination. $U \in G_i$ stands for the event that the observed particles in

U have all been obtained by random deformation of elements of G_i. A random vector $\vec{Q}$ of measurements is carried out and since the probability laws of $\vec{Q}|U \in G_1$ and of $\vec{Q}|U \in G_2$ are supposed to be different this leads usually to different likelihoods $L(G_1 : \vec{q})$ or $L(G_2 : \vec{q})$ of the observations $\vec{Q} = \vec{q}$ when the particles originated from G_1 or from G_2 respectively.
Furthermore discrimination will also be based on the a priori probabilities p_i of the event $U \in G_i$ (where of course $p_1 + p_2 = 1$ is satisfied) and on the costs $C(1|2)$ respectively $C(2|1)$ of deciding incorrectly that U comes from G_1 respectively of deciding incorrectly that U comes from G_2 provided these quantities are known.
An identification rule is characterized by the indication of a subset $\mathcal{X}_2$ in the sample space of $\vec{Q}$ with the specification that U is attributed to the population G_2 if and only if the realized value $\vec{q}$ of $\vec{Q}$ lies in $\mathcal{X}_2$. In such situations the optimal Bayes identification rule, which minimizes the overall mean cost of incorrect shape-type attribution, consists in deciding that U comes from G_2 if and only if the condition
$\vec{q} \in \{C(2|1)p_1L(G_1 : \vec{q}) < C(1|2)p_2L(G_2 : \vec{q})\}$
is fulfilled (Anderson,1984, Chapter 6). If either the a priori probabilities p_1 and p_2 or the costs of incorrect classification $C(2|1)$ and $C(1|2)$ or both are actually unknown the best identification rule is obtained by deleting these quantities in the above inequality.

ON RANDOMLY DEFORMED PARTICLES

Since most applications concern sets in dimension 2 and 3, our considerations will be restricted to planar particles in $\mathcal{R}^2$ expressed in Cartesian (x,y)-coordinates or spatial particles in $\mathcal{R}^3$ expressed in Cartesian (x,y,z)-coordinates.

When the population G_1 is the correct reference set we specify the planar prototype to be rectangular and the spatial prototype to be a circular cylinder; under this hypothesis the particles have thus roughly the form of sticks. We are here concerned with experimental set-ups where the original shape is completely specified by a symmetry axis and by the cross-sectional breadth measured at the points along this symmetry axis, a set-description model proposed in (Stoyan and Stoyan, 1992, pp.81 and 82). Note that in this paper the prototype shape is described by the thickness of the body measured orthogonally to the symmetry axis and not by the radius vector function as in (Streit, 1997, 2000, 2003, 2005, 2006). It is supposed that the symmetry axis is recognizable either by means of landmarks or by taking into account a geometrical property of the particle (as for instance that the symmetry axis coincides with the axis of the diameter, thus the segment of maximal length realized within the body). The particles chosen for the population G_1 are particularly simply structered in view of the fact that the cross-sectional breadth does not change along this axis while it is positive. On the other hand we assume that under G_2 we have a shape-type which allows changes in cross-sectional breadth. At level y respectively z on the symmetry axis the cross-sectional breadth is equal $a > 0$ in the population G_1 and equal $a \cdot b(y)$ respectively $a \cdot b(z)$ in the population G_2, where the function b is not identically equal to 1, serves to model the variation of the cross-sectional breadth along the symmetry-axis and takes only positive values on the relevant segment of this axis.

In our stochastic models certain features like the position, the size or the orientation of the particles are supposed independent of the shape and need therefore not to be taken into consideration. Each of the observed particles may thus be first reoriented and standardized by putting its axis of symmetry in vertical position and by assigning the ordinate $y = -1/2$ to its lowest point and the ordinate $y = 1/2$ to its top in the planar case and proceeding analogously with the assignment of the z-values in the spatial case. This standardization is carried out by using a unit of length in all length measurements for the same particle adjusted to produce this situation. For a fixed integer $n \in \{2, 3, ...\}$ the cross-sectional breadths orthogonally to this axis are measured at the ordinate levels $y = -(n-1)/(2n), y = -(n-2)/(2n), \ldots, y = (n-2)/(2n), y = (n-1)/(2n)$ in the planar case, with y replaced by z and taking the straight line orthogonal to the z - axis which yields the largest value in the spatial case. The prototype shape used are chosen in such a way that any of these level straight lines cuts the boundary of the prototype shape only in two points symmetrically arranged around the symmetry axis. n represents somehow the degree of measuring-effort undertaken per individual particle. Let N be the number of observed particles and $\vec{Q} = (Q^{(l)}(i/(2n))[i = -(n-1), \ldots, n-1; l = 1, \ldots, N])'$ the set of measurements to be taken, where $Q^{(l)}(i/(2n))$ designates the cross-sectional breadth (i.e. maximal thickness) at level $y = i/(2n)$ or $z = i/(2n)$ of the l-the particle, $q^{(l)}(i/(2n))$ its realized value and $'$ the transposition of a matrix.

Since our data set consists simply of an ordered set of length measurements it is sufficient to explain the effect of random deformation only at the ordinate levels where such mesurements are taken. We shall here assume that random deformation of a particle is caused by an individual dilatation of the cross-sectional breadth of each particle at each level of the ordinate.

Thus random deformation is described by the following relations between the random variables of the competing stochastic models valid for the population G_1 respectively G_2:
For population $G_1 : Q^{(l)}(i/(2n)) = Y^{(l)}(i/(2n)) \cdot a$
$[i = -(n-1), \ldots, (n-1); l = 1, \ldots, N]$
and
for population $G_2 : Q^{(l)}(i/(2n)) = Y^{(l)}(i/(2n)) \cdot a \cdot b(i/(2n))$
$[i = -(n-1), \ldots, (n-1); l = 1, \ldots, N]$.
For $l \in 1, \ldots, N$ $\quad (Y^{(l)}(i/(2n))\,[i = -(n-1), \ldots, n-1])'$ are supposed to be independent random samples from an exponential distribution with parameter λ.
Taking into account that the prototype shapes of the populations G_1 and G_2 should give rise to the same observed particles, it is reasonable that the following additional condition is fulfilled, whenever the family of shapes admitted in the population G_2 is indexed by at least two unrelated parameters specifying an individual figure within the shape-class:
The prototype shape of the population G_1 has in the planar case the same surface area as the prototype shape of G_2 and the prototype shape in the population G_1 has in the spatial case the same volume as the prototype shape of G_2.

DETERMINATION OF THE DISCRIMINANT RULE FOR SHAPE IDENTIFICATION.

We shall now determine the identification rule which allows to decide if it is appropriate to adopt the shape of a stick (rectangular or cylindric) or whether in view of the measurements we should rather opt for the alternative shape-type of G_2 involving the function b not identically equal to 1. The measured lengths do not have the same chance to arise from G_1 or from G_2 and this fact will lead us to decide which model we should prefer. We have first to evaluate the likelihood functions. We find
$L(G_1, \lambda, a : \vec{Q} = \vec{q}) = (\lambda/a)^{(2n-1)N} \prod_{l=1}^{N} \prod_{i=-(n-1)}^{n-1} \exp[-(\lambda q^{(l)}(i/(2n)))/a]$
and
$L(G_2, \lambda, a : \vec{Q} = \vec{q}) = (\lambda/a)^{(2n-1)N} \prod_{i=-(n-1)}^{n-1} [(1/b(i/(2n)))^N \cdot$
$\prod_{l=1}^{N} \exp[-(\lambda q^{(l)}(i/(2n)))/(ab(i/(2n)))]]$.
The Bayes identification rule which minimizes the overall mean cost of incorrect shape attribution consists in deciding that U comes from G_2 if the condition
$\vec{q} \in \{C(2|1)p_1 L(G_1, \lambda, a : \vec{Q} = \vec{q}) < C(1|2)p_2 L(G_2, \lambda, a : \vec{Q} = \vec{q})\}$
is satisfied. The above condition is equivalent to

$\vec{q} \in \{C(2|1)p_1/(C(1/2)p_2) <$
$\prod_{i=-(n-1)}^{n-1} b(i/(2n))^{-N} \cdot \exp[(\lambda/a) \sum_{l=1}^{N} \sum_{i=-(n-1)}^{n-1} q^{(l)}(i/(2n))(1-1/b(i/(2n)))]\}$.
Thus we assign the set U with measurements $\vec{Q} = \vec{q}$ to the population G_2 if and only if
$\ln((C(2|1)p_1)/(C(1|2)p_2)) + N \sum_{i=-(n-1)}^{n-1} \ln(b(i/(2n))) <$
$(\lambda/a) \sum_{l=1}^{N} \sum_{i=-(n-1)}^{n-1} [q^{(l)}(i/(2n))(1 - 1/b(i/(2n)))] =: t_{N,n}$.
If $p_1, p_2, C(1|2)$ and $C(2|1)$ are not known, then the expression
$\ln((C(2|1)p_1)/(C(1|2)p_2))$
has to be dropped on the left hand side of the inequality, yielding the likelihood identification rule with inequality
$N \sum_{i=-(n-1)}^{n-1} \ln(b(i/(2n))) < t_{N,n}$.

PROPERTIES OF THIS IDENTIFICATION RULE

Designating by $T_{N,n}$ the random variable corresponding to $t_{N,n}$, we note that $T_{N,n} = \sum_{i=1}^{N} T_{N,n}^{(l)}$ is the sum of N independent identically distributed random variables
$T_{N,n}^{(l)} = (\lambda/a) \sum_{i=-(n-1)}^{n-1} Q^{(l)}(i/(2n))(1 - 1/b(i/(2n)))$.
If the particles come from G_1 we have
$m_{N,n,G_1} := E[T_{N,n} : G_1] = N \sum_{i=-(n-1)}^{n-1} (1 - 1/b(i/(2n)))$
and
$v_{N,n,G_1} := Var[T_{N,n} : G_1] = N \sum_{i=-(n-1)}^{n-1} (1 - 1/b(i/(2n)))^2$.
According to the central limit theorem, we have for large N (n being fixed) $T_{N,n} \sim N(m_{N,n,G_1}, v_{N,n,G_1})$, where N(g,h) designates the normal distribution with mathematical expectation g and variance h .
If the particles come from G_2 we have
$m_{N,n,G_2} := E[T_{N,n} : G_2] = N \sum_{i=-(n-1)}^{n-1} (b(i/(2n)) - 1)$
and
$v_{N,n,G_2} := Var[T_{N,n} : G_2] = N \sum_{i=-(n-1)}^{n-1} (b(i/(2n)) - 1)^2$.
In this case also $T_{N,n}$ follows for large N an asymptotic normal distribution namely
$T_{N,n} \sim N(m_{N,n,G_2}, v_{N,n,G_2})$.
Based on these findings we are able to evaluate the asymptotic probabilities of misclassification errors which are below given for the likelihood identification rule.
The probability of an incorrect decision, stating that G_2 is the population from which originated the observed particles, equals for large N
$P(T_{N,n} > N \sum_{i=-(n-1)}^{n-1} b(i/(2n)) : G_1) = 1 - \Phi([N \sum_{i=-(n-1)}^{n-1} \ln(b(i/(2n))) - m_{N,n,G_1}]/\sqrt{v_{N,n,G_1}})$,

where Φ designates the distribution function of the standard normal distribution.

The probability of an incorrect classification of the observed particles to G_1 is for large N given by $P(T_{N,n} \leq N\sum_{i=-(n-1)}^{n-1} b(i/(2n)) : G_2) = \Phi([N\sum_{i=-(n-1)}^{n-1} \ln(b(i/(2n))) - m_{N,n,G_2}]/\sqrt{v_{N,n,G_2}})$.

We have for positive b-values $\ln(b(i/(2n))) = \ln(1+(b(i/(2n))-1)) > (b(i/(2n))-1)/b(i/(2n))$. Therefore in the formula for the misclassification probability to G_2 the argument of Φ is a positive number and thus this probability is smaller than 1/2 and tends to O for $N \to \infty$.

On the other hand $\ln(b(i/(2n))) = \ln(1+(b(i/(2n))-1)) < b(i/(2n))-1$ for positive b-values and thus in the formula for the misclassification probability to G_1 the argument of Φ is a negative number and therefore this probability is also smaller than 1/2 and tends to 0 for $N \to \infty$.

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